Introduction to Quantum Information Processing
COM 309 Week 2

Exercise 1

Properties of Pauli matrices

We collect useful properties of Pauli matrices. Let $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ a vector formed by the 3 Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The identity matrix is denoted $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

1. Show that all 2×2 matrices, A, can be written as a linear combination of I and σ_x, σ_y, σ_z:

   $$A = a_0 I + a_1 \sigma_x + a_2 \sigma_y + a_3 \sigma_z.$$

   This can also be written as $A = a_0 I + \vec{a} \cdot \vec{\sigma}$ where $\vec{a} \cdot \vec{\sigma}$ is an "inner product" between the "vectors" $\vec{a} = (a_1, a_2, a_3)$ et $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$.

   Check also that if $A = A^\dagger$ we have a_0, a_1, a_2, $a_3 \in \mathbb{R}$.

2. Check the following algebraic identities:

   $$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$$
   $$\sigma_x \sigma_y = i\sigma_z$$
   $$\sigma_y \sigma_z = i\sigma_x$$
   $$\sigma_z \sigma_x = i\sigma_y$$

   Deduce

   $$\sigma_x \sigma_y + \sigma_y \sigma_x = 0$$
   $$\sigma_y \sigma_z + \sigma_z \sigma_y = 0$$
   $$\sigma_z \sigma_x + \sigma_x \sigma_z = 0$$

3. Let $[A, B] = AB - BA$ be the "commutator". Show (you may use preceding results)

   $$[\sigma_x, \sigma_y] = 2i\sigma_z$$
   $$[\sigma_y, \sigma_z] = 2i\sigma_x$$
   $$[\sigma_z, \sigma_x] = 2i\sigma_y$$

These relations are called "commutation relations for spin".

4. Compute eigenvalues and eigenvectors of σ_x, σ_y, σ_z. Check that the eigenvalues satisfy $\mathrm{Tr}\,\sigma_x = \mathrm{Tr}\,\sigma_y = \mathrm{Tr}\,\sigma_z = 0$ et $\det \sigma_x = \det \sigma_y = \det \sigma_z = -1$.

5. Dirac notation: set
$$|{\uparrow}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ et } |{\downarrow}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
Check that
$$\sigma_z = |{\uparrow}\rangle\langle{\uparrow}| - |{\downarrow}\rangle\langle{\downarrow}|$$
$$\sigma_x = |{\uparrow}\rangle\langle{\downarrow}| + |{\downarrow}\rangle\langle{\uparrow}|$$
$$\sigma_y = i\,|{\downarrow}\rangle\langle{\uparrow}| - i\,|{\uparrow}\rangle\langle{\downarrow}|$$

Exercise 2

Exponentials of Pauli matrices

1. We define the exponential of a matrix A by (for $t \in \mathbb{R}$)
$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + ...$$
We want to prove the identity:
$$e^{it\vec{n}\cdot\vec{\sigma}} = I\cos t + i\vec{n}\cdot\vec{\sigma}\sin t$$
where $\vec{n}$ is a unit vector and $t \in \mathbb{R}$. Remark that this is a generalization of Euler's identity:
$$e^{i\theta} = \cos\theta + i\sin\theta$$
To show the identity show first that:
$$(\vec{n}\cdot\vec{\sigma})^2 = I$$
Use Taylor expansions of $\cos t$ and $\sin t$ to deduce the wanted identity above.

2. Explicitly write 2×2 matrices (in component/array form) $\exp(it\sigma_x)$; $\exp(it\sigma_y)$; $\exp(it\sigma_z)$ as well as $\exp(it\vec{n}\cdot\vec{\sigma})$.

Exercise 3

Inner products of tensor products

1. Which pairs of vectors are mutually orthogonal? Work in Dirac notation.

$$|0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$\left(\frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle\right)$$

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \left(\frac{2}{\sqrt{5}}|0\rangle - \frac{1}{\sqrt{5}}|1\rangle\right)$$

2. Given the canonical coordinate representation for the states $|0\rangle$ and $|1\rangle$ compute the component form (4 components of the above tensor products).

3. Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 5 \\ 8 & 10 \end{pmatrix}$$

Express the matrices in Dirac notation in the $|0\rangle, |1\rangle$ basis. Compute their tensor products $A \otimes B$ and $B \otimes A$ in Dirac notation.

4. Make the calculation of $A \otimes B$ and $B \otimes A$ in component form in the canonical basis. Check that the result is consistent with the one obtained above in Dirac notation.

Exercise 4

Product versus entangled states

Prove whether the following states are product or entangled states ? (check also they are correctly normalized)

1. $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

2. $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$

3. $\frac{1}{\sqrt{6}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{6}|10\rangle - \frac{1}{\sqrt{3}}|11\rangle$

4. $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $|\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, $|\beta_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$, $|\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, .

5. $\frac{1}{\sqrt{1+\epsilon^2}}(|00\rangle + \epsilon|11\rangle)$, for $0 \leq \epsilon \leq 1$

6. $\frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$

7. $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$

8. $\frac{1}{2\sqrt{2}}(|000\rangle + |100\rangle + |010\rangle + |001\rangle + |110\rangle + |101\rangle + |011\rangle + |111\rangle)$

Exercise 5

Unitary transformations

Verify that the following transformations are unitary (check also the identities between matrix tables and Dirac notation):

1. Simple time evolution of the type $|\psi_t\rangle = e^{i\omega t}|\psi_0\rangle$. This is for example the time evolution of a free photon of frequency $\nu = \omega/2\pi$ or energy $E = h\nu = \hbar\omega$.

2. The Hadamard gate.

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|) \tag{1}$$

   Check how the basis $|0\rangle$, $|1\rangle$ is transformed. Remark: in interferometers models for example a semi-transparent mirror.

3. The X or NOT gate

$$X = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|$$

   Check how the basis $|0\rangle$, $|1\rangle$ is transformed. Remark: in interfreometers it models for example a reflecting mirror.

4. $U_1 \otimes U_2$ if U_1 and U_2 are unitary. Remark: if U_i, $i = 1, 2$ act each on a one-qubit Hilbert space $\mathbb{C}^2$ then the tensor product acts on the two-qubit space $\mathbb{C}^2 \otimes \mathbb{C}^2$.

5. The control-NOT gate. This gate flips the control bit (the second) if the target bit (the first) is 1.

$$\begin{aligned} CNOT &= \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix} = (|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|) \\ &= (|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X) \end{aligned}$$

Compute the following states:

$$|\beta_{ij}\rangle = CNOT \otimes H|i\rangle \otimes |j\rangle, \quad i, j = 0, 1$$

and show that they are equal to the four Bell states introduced in class. Deduce that $\{|\beta_{ij}\rangle, i, j = 0, 1\}$ is an orthonormal basis. This identity shows that $CNOT$ entangles the two qubits.