Problem Set 2

Problem 1: Axioms

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Using only the axioms given in the definition of a probability measure, namely:

(i) $\mathbb{P}(\emptyset) = 0\,,\ \mathbb{P}(\Omega) = 1\,;$

(ii) If $(A_n,\, n \geq 1)$ is a sequence of *disjoint* events in $\mathcal{F}$, then $\mathbb{P}(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)\,;$

show the properties below.

Important note: For parts c), d) and e), induction does *not* work! You need to show that each property holds for an infinite number of events at once, using the above axiom (ii).

a) If $A, B \in \mathcal{F}$ and $A \subset B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$ and $\mathbb{P}(B \backslash A) = \mathbb{P}(B) - \mathbb{P}(A)$. Also, $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

b) If $A, B \in \mathcal{F}$, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

c) If $(A_n,\, n \geq 1)$ is a sequence of events in $\mathcal{F}$, then $\mathbb{P}(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mathbb{P}(A_n)$.

d) If $(A_n,\, n \geq 1)$ is a sequence of events in $\mathcal{F}$ such that $A_n \subset A_{n+1}$, $\forall n \geq 1$, then $\mathbb{P}(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mathbb{P}(A_n)$.

e) If $(A_n,\, n \geq 1)$ is a sequence of events in $\mathcal{F}$ such that $A_n \supset A_{n+1}$, $\forall n \geq 1$, then $\mathbb{P}(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mathbb{P}(A_n)$.

Problem 2: CDF

a) Which of the following are cdfs?

1. $F_1(t) = \exp(-e^{-t})$, $t \in \mathbb{R}$
2. $F_2(t) = \frac{1}{1-e^{-t}}$, $t \in \mathbb{R}$
3. $F_3(t) = 1 - \exp(-1/|t|)$, $t \in \mathbb{R}$
4. $F_4(t) = 1 - \exp(-e^{t})$, $t \in \mathbb{R}$

b) Let now F be a generic cdf. Which of the following functions are guaranteed to be also cdfs?

5. $F_5(t) = F(t^2)$, $t \in \mathbb{R}$
6. $F_6(t) = F(t)^2$, $t \in \mathbb{R}$
7. $F_7(t) = F(1 - \exp(-t))$, $t \in \mathbb{R}$
8. $F_8(t) = \begin{cases} 1 - \exp(-F(t)/(1-F(t))) & \text{if } F(t) < 1 \\ 1 & \text{if } F(t) = 1 \end{cases} \quad t \in \mathbb{R}$

Problem 3: Jumps of the CDF

Let X be an arbitrary random variable and $\mathbb{F}_X$ its CDF. Show that $\mathbb{F}_X$ can have at most a discrete number of jumps.

Problem 4: Sigma field generated by random variables

a) Let X, Y be two random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G} = \sigma(X) \cap \sigma(Y)$ [fact: it can be shown that $\mathcal{G}$ is a σ-field]. Is it true that $\{X \leq Y\} \in \mathcal{G}$?

b) Let X, Y be two independent random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Is it always true that $\sigma(X + Y) = \sigma(X, Y)$?

c) Let X be a continuous random variable whose pdf p_X is a continuous function on $\mathbb{R}$.
Let now $Y = X^2$. Is it always true that the pdf p_Y is also a continuous function on $\mathbb{R}$?

d) Let F be a generic cdf. Is it always true that the function $G : \mathbb{R} \to [0, 1]$ defined as

$$G(t) = F(t^3 + 3t^2 + 3t + 1), \quad t \in \mathbb{R}$$

is also a cdf?