Solutions to Homework 2

Exercise 1.

a) 1. true, 2. true, 3. false, 4. true b) 5. false, 6. true, 7. false, 8. true.

Exercise 2.

a) Use $B = A \cup (B \backslash A)$, where A and $B \backslash A$ are disjoint, as well as $\Omega = A \cup A^c$ and $\mathbb{P}(\Omega) = 1$.

b) Use $A \cup B = A \cup (B \backslash (A \cap B))$ where A and $B \backslash (A \cap B)$ are disjoint, as well as a).

c) Use $\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} B_n$, where $B_n = A_n \backslash (A_1 \cup \ldots \cup A_{n-1})$; the B_n are disjoint, so by axiom (ii) and a),

$$\mathbb{P}(\cup_{n=1}^{\infty} A_n) = \mathbb{P}(\cup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mathbb{P}(B_n) \leq \sum_{n=1}^{\infty} \mathbb{P}(A_n)$$

d) $\mathbb{P}(\cup_{n\geq 1} A_n) = \mathbb{P}(\cup_{n\geq 1}(A_n \cap A_{n-1}^c)) \overset{(*)}{=} \sum_{n=1}^{\infty} \mathbb{P}(A_n \cap A_{n-1}^c) = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{P}(A_i \cap A_{i-1}^c)$ $\overset{(**)}{=} \lim_{n\to\infty} \mathbb{P}(\cup_{i=1}^{n}(A_i \cap A_{i-1}^c)) = \lim_{n\to\infty} \mathbb{P}(\cup_{i=1}^{n} A_i) = \lim_{n\to\infty} \mathbb{P}(A_n)$, where $(*)$, $(**)$ follow from the fact that the sets $A_n \cap A_{n-1}^c$ are disjoint.

e) Using parts a) and d): $\mathbb{P}(\cap_{n\geq 1} A_n) = 1 - \mathbb{P}((\cap_{n\geq 1} A_n)^c) = 1 - \mathbb{P}(\cup_{n\geq 1} A_n^c) = 1 - \lim_{n\to\infty} \mathbb{P}(A_n^c) = \lim_{n\to\infty} \mathbb{P}(A_n)$.

Exercise 3.*

a) First, note that the range of the random variable X is $[0,1]$. Thus, the CDF $F_X(t) = 0$ for $t < 0$ and $F_Y(t) = 1$ for $t \geq 1$.

Now, for $t \in [0,1]$, we have:

$$\begin{aligned} F_X(t) = \mu_X((-\infty, t]) &= \mathbb{P}(\{(\omega_1, \omega_2) \in [0,1]^2 : X(\omega_1, \omega_2) \leq t\}) \\ &= \mathbb{P}(\{(\omega_1, \omega_2) \in [0,1]^2 : \omega_1\omega_2 \leq t\}) \end{aligned}$$

Now, one could just compute the probability by integrating the area under the curve $\omega_1\omega_2 \leq t$ that lies within $[0,1] \times [0,1]$ as follows:

$$\begin{aligned} \mathbb{P}(\{(\omega_1, \omega_2) \in [0,1]^2 : \omega_1\omega_2 \leq t\}) &= t + \int_t^1 \frac{t}{\omega_1} d\omega_1 \\ &= t(1 - \ln t) \end{aligned}$$

Therefore,

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0, \\ t(1 - \ln t) & \text{if } t \in [0,1] \\ 1 & \text{if } t > 1 \end{cases}$$

b) First, note that the range of the random variable Y is $\Big[-\frac{1}{2}, \frac{1}{2}\Big]$. Thus, the CDF $F_Y(t) = 0$ for $t < -\frac{1}{2}$ and $F_Y(t) = 1$ for $t \geq \frac{1}{2}$.

Now, for $t \in \Big[-\frac{1}{2}, \frac{1}{2}\Big]$, we have:

$$\begin{aligned} F_Y(t) = \mu_Y((-\infty, t]) &= \mathbb{P}(\{(\omega_1, \omega_2) \in [0,1]^2 : Y(\omega_1, \omega_2) \leq t\}) \\ &= \mathbb{P}(\{(\omega_1, \omega_2) \in [0,1]^2 : \omega_1 - \omega_2 \leq 2t\}) \end{aligned}$$

Note that the area $\{(\omega_1, \omega_2) \in [0,1]^2 : \omega_1 - \omega_2 \leq 2t\}$ represents different shapes in $[0,1] \times [0,1]$ for positive and negative values of $2t$. Thus, we divide our analysis into two cases:

Case 1: $-\frac{1}{2} < t \leq 0$:

The area $\{(\omega_1, \omega_2) \in [0,1]^2 : \omega_1 - \omega_2 \leq 2t\}$ represents a right-angled triangle (Δ_1) is an element of the sigma field $\mathcal{F} = \mathcal{B}([0,1]^2)$. Thus, the probability measure $\mathbb{P}(\Delta_1)$ is given by its area. Thus,

$$F_Y(t) = Area(\Delta_1) = \frac{1}{2}(1+2t)(1+2t)$$

Case 2: $0 < t \leq \frac{1}{2}$:

The area $\{(\omega_1, \omega_2) \in [0,1]^2 : \omega_1 - \omega_2 \leq 2t\}$ represents a pentagon (Δ_2) in this case which is again an element of the sigma field $\mathcal{F} = \mathcal{B}([0,1]^2)$. Thus, the probability measure $\mathbb{P}(\Delta_2)$ is given by its area which can be easily computed as:

$$F_Y(t) = Area(\Delta_2) = 1 - \frac{1}{2}(1-2t)(1-2t)$$

Thus, the CDF of the random variable Y is the following:

$$F_Y(t) = \begin{cases} 0 & \text{if } t \leq -\frac{1}{2}, \\ \frac{1}{2}(1+2t)^2 & \text{if } -\frac{1}{2} < t \leq 0 \\ 1 - \frac{1}{2}(1-2t)^2 & \text{if } 0 < t \leq \frac{1}{2} \\ 1 & \text{if } t > \frac{1}{2} \end{cases}$$

Exercise 4.

a) We have

$$\begin{aligned} \mathbb{P}(\{Y_n \leq t\}) = 1 - \mathbb{P}(\{Y_n > t\}) &= 1 - \mathbb{P}(\{\min\{X_1, \ldots, X_n\} > t\}) = 1 - \mathbb{P}(\cap_{j=1}^n \{X_j > t\}) \\ &= 1 - \textstyle\prod_{j=1}^n \mathbb{P}(\{X_j > t\}) = 1 - \mathbb{P}(\{X_1 > t\})^n \end{aligned}$$

where the last two equalities follow from the assumption that the X's are i.i.d. Therefore,

$$\mathbb{P}(\{Y_n \leq t\}) = 1 - (\exp(-t))^n = 1 - \exp(-nt)$$

b) Under the assumptions made, n is large and t is such that $nt \ll 1$, so using Taylor's expansion $\exp(-x) \simeq 1 - x$, we obtain

$$\mathbb{P}(\{Y_n \leq t\}) \simeq 1 - (1 - nt) = nt \quad \text{while} \quad \mathbb{P}(\{X_1 \leq t\}) = 1 - \exp(-t) \simeq t$$

and therefore $\mathbb{P}(\{Y_n \leq t\}) \simeq n\,\mathbb{P}(\{X_1 \leq t\})$.

c) We have similarly

$$\mathbb{P}(\{Z_n \geq t\}) = 1 - \mathbb{P}(\{Z_n < t\}) = 1 - \mathbb{P}(\{\max\{X_1, \ldots, X_n\} < t\}) = 1 - \mathbb{P}(\cap_{j=1}^n \{X_j < t\})$$
$$= 1 - \prod_{j=1}^n \mathbb{P}(\{X_j < t\}) = 1 - \mathbb{P}(\{X_1 < t\})^n = 1 - (1 - \exp(-t))^n$$

d) Under the assumptions made, n is large and t is such that $n\exp(-t) \ll 1$, so using again the same Taylor expansion as above, we obtain

$$\mathbb{P}(\{Z_n \geq t\}) \simeq 1 - (1 - n\,\exp(-t)) = n\,\exp(-t) \quad \text{while} \quad \mathbb{P}(\{X_1 \geq t\}) = \exp(-t)$$

and therefore $\mathbb{P}(\{Z_n \geq t\}) \simeq n\,\mathbb{P}(\{X_1 \geq t\})$.

Exercise 5.

a) Yes. Here, we need to check that $\forall\ B \in \mathcal{B}(\mathbb{R})$, we have $X^{-1}(B) \in \mathcal{F}$. Specifically, check for $B = \{0\}, \{-2\}, \{1\}, \{0,1\}, \{0,-2\}, \{1,-2\}$.

b) No. Here, we have $X^{-1}(-1) = \{a,b\}$, $X^{-1}(1) = \{c\}$, $X^{-1}(2) = \{d\}$. However, $\{c\}, \{d\} \notin A$. Thus, X is not a $\mathcal{F}-$measurable random variable.

c) Let us begin by recalling that the cdf is a non-decreasing and a right-continuous function i.e., $\forall m \in \mathbb{R}$, we have:

$$\lim_{\epsilon \to 0} F(m - \epsilon) \leq F(m) = \lim_{\epsilon \to 0} F(m + \epsilon)$$

Now, defining $m_0 := \sup\{x \in \mathbb{R} : F(x) < 1/2\}$. Then, $F(m_0) \geq 1/2$. How??. Assume, it's not true i.e., $F(m_0) < 1/2$. Since, the cdf F is a right continuous function, it implies that $\exists\ m_0' > m_0$ such that $F(m_0') < 1/2$. However, this violates our definition of m_0. Thus, our assumption is wrong and consequently $F(m_0) \geq 1/2$. Furthermore, $\lim_{\epsilon \to 0} F(m_0 - \epsilon) \leq 1/2$. Hence, m_0 is the (smallest) median of X.

Let's also define $m_1 := \sup\{x \in \mathbb{R} : F(x) \leq 1/2\}$. Then, $F(m_1) \geq 1/2$ as F is a non-decreasing function. Furthermore, $\lim_{\epsilon \to 0} F(m_1 - \epsilon) \leq 1/2$. Thus, m_1 is the (largest) median of X.

Now, note that $\{x \in \mathbb{R} : F(x) < 1/2\} \subseteq \{x \in \mathbb{R} : F(x) \leq 1/2\}$. Thus, $m_0 \leq m_1$. We can now show that the closed interval $[m_0, m_1]$ contains the medians i.e., $\forall\ m \in [m_0, m_1]$, m is a median of X. Since, $m_0 \leq m \leq m_1$, it's not hard to see that $1/2 \leq F(m_0) \leq F(m) \leq F(m_1)$. Furthermore, note that $\forall \epsilon > 0$, $\lim_{\epsilon \to 0} F(m_0 - \epsilon) \leq \lim_{\epsilon \to 0} F(m - \epsilon) \leq \lim_{\epsilon \to 0} F(m_1 - \epsilon) \leq 1/2$. Thus, we have that $\forall m \in [m_0, m_1]$, $\lim_{\epsilon \to 0} F(m - \epsilon) \leq \frac{1}{2} \leq F(m)$.

The median of X is unique when the cdf of X is a strictly increasing function. In this case, the closed interval $[m_0, m_1]$ reduces to a singleton set.