Homework 2

Exercise 1.

a) Which of the following are cdfs?

1. $F_1(t) = \exp(-e^{-t})$, $t \in \mathbb{R}$
2. $F_2(t) = \frac{e^t}{e^t+e^{-t}}$, $t \in \mathbb{R}$
3. $F_3(t) = 1 - \exp(-1/|t|)$, $t \in \mathbb{R}$
4. $F_4(t) = 1 - \exp(-e^t)$, $t \in \mathbb{R}$

b) Let now F be a generic cdf. Which of the following functions are guaranteed to be also cdfs?

5. $F_5(t) = F(t^2)$, $t \in \mathbb{R}$
6. $F_6(t) = F(t)^2$, $t \in \mathbb{R}$
7. $F_7(t) = F(1 - \exp(-t))$, $t \in \mathbb{R}$
8. $F_8(t) = \begin{cases} 1 - \exp(-F(t)/(1 - F(t))) & \text{if } F(t) < 1 \\ 1 & \text{if } F(t) = 1 \end{cases} \quad t \in \mathbb{R}$

Exercise 2.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Using only the axioms given in the definition of a probability measure, show the following properties:

a) If $A, B \in \mathcal{F}$ and $A \subset B$, then $\mathbb{P}(A) \le \mathbb{P}(B)$ and $\mathbb{P}(B\backslash A) = \mathbb{P}(B) - \mathbb{P}(A)$. Also, $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

b) **[inclusion-exclusion formula]** If $A, B \in \mathcal{F}$, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

c) **[union bound]** If $(A_n,\, n \ge 1)$ is a sequence of events in $\mathcal{F}$, then $\mathbb{P}(\cup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} \mathbb{P}(A_n)$.

d) If $(A_n,\, n \ge 1)$ is a sequence of events in $\mathcal{F}$ such that $A_n \subset A_{n+1}$, $\forall n \ge 1$, then $\mathbb{P}(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mathbb{P}(A_n)$.

e) If $(A_n,\, n \ge 1)$ is a sequence of events in $\mathcal{F}$ such that $A_n \supset A_{n+1}$, $\forall n \ge 1$, then $\mathbb{P}(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mathbb{P}(A_n)$.

Exercise 3.*

Let $\Omega = [0,1]^2$, $\mathcal{F} = \mathcal{B}([0,1]^2)$ and $\mathbb{P}$ be the probability measure defined on $(\Omega, \mathcal{F})$ defined as

$$\mathbb{P}(\,]a,b[\times]c,d[\,) = (b-a)\cdot(d-c), \quad \text{for } 0 \le a < b \le 1 \text{ and } 0 \le c < d \le 1$$

which can be extended uniquely to all Borel sets in $\mathcal{B}([0,1]^2)$, according to Caratheodory's extension theorem.

Let us now consider the following two random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$:

$$X(\omega_1, \omega_2) = \omega_1\omega_2 \quad \text{and} \quad Y(\omega_1, \omega_2) = \frac{\omega_1 - \omega_2}{2}$$

a) Compute and plot the cdf F_X of X.
b) Compute and plot the cdf F_Y of Y.

Exercise 4.

Let $X_1, \dots, X_n$ be i.i.d. $\sim \mathcal{E}(1)$ random variables (i.e., they are independent and identically distributed, all with the exponential distribution of parameter $\lambda = 1$).

a) Compute the cdf of $Y_n = \min\{X_1, \dots, X_n\}$.

b) How do $\mathbb{P}(\{Y_n \leq t\})$ and $\mathbb{P}(\{X_1 \leq t\})$ compare when n is large and t is such that $t \ll \frac{1}{n}$?

c) Compute the cdf of $Z_n = \max\{X_1, \dots, X_n\}$.

d) How do $\mathbb{P}(\{Z_n \geq t\})$ and $\mathbb{P}(\{X_1 \geq t\})$ compare when n is large and t is such that $t \gg \log(n)$?

Exercise 5.

a) Let $(\Omega, \mathcal{F})$ denote a measurable space. Define a function $X : \Omega \to \mathbb{R}$ as follows:

$$X(\omega) = \begin{cases} -2 & \text{on } F_1 \\ 1 & \text{on } F_1^c \cup F_2 \\ 0 & \text{on } F_1^c \cup F_2^c \end{cases}$$

where F_1, $F_2 \in \mathcal{F}$. Is X a random variable on the measurable space $(\Omega, \mathcal{F})$?

b) Let $(\Omega, \mathcal{F})$ denote a measurable space such that $\Omega = \{a, b, c, d\}$ and $\mathcal{F} = \{\phi, \{a, b\}, \{c, d\}, \Omega\}$. Let X be a function such that $X(a) = X(b) = -1$, $X(c) = 1$ and $X(d) = 2$. Is X a $\mathcal{F}-$ measurable random variable ?

c) [**Median of a random variable**] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X be a $\mathcal{F}-$ measurable random variable. Then, a real number m is called the median of X whenever $\lim_{\epsilon \to 0} F(m - \epsilon) \leq \frac{1}{2} \leq F(m)$, $(\epsilon > 0)$ where F denotes the cdf of X. Show that every random variable X has at least one median, and that the set of medians of X is a closed interval of $\mathbb{R}$. Under which condition, is the median of a random variable unique?