Ecole Polytechnique Fédérale, Lausanne
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Solution Set 2

Problem 1: Axioms

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Using only the axioms given in the definition of a probability measure, namely:

(i) $\mathbb{P}(\emptyset) = 0\,,\ \mathbb{P}(\Omega) = 1\,;$

(ii) If $(A_n,\, n \geq 1)$ is a sequence of *disjoint* events in $\mathcal{F}$, then $\mathbb{P}(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)\,;$

show the properties below.

Important note: For parts c), d) and e), induction does *not* work! You need to show that each property holds for an infinite number of events at once, using the above axiom (ii).

a) If $A, B \in \mathcal{F}$ and $A \subset B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$ and $\mathbb{P}(B \backslash A) = \mathbb{P}(B) - \mathbb{P}(A)$. Also, $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

b) If $A, B \in \mathcal{F}$, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

c) If $(A_n,\, n \geq 1)$ is a sequence of events in $\mathcal{F}$, then $\mathbb{P}(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mathbb{P}(A_n)$.

d) If $(A_n,\, n \geq 1)$ is a sequence of events in $\mathcal{F}$ such that $A_n \subset A_{n+1}$, $\forall n \geq 1$, then $\mathbb{P}(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mathbb{P}(A_n)$.

e) If $(A_n,\, n \geq 1)$ is a sequence of events in $\mathcal{F}$ such that $A_n \supset A_{n+1}$, $\forall n \geq 1$, then $\mathbb{P}(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mathbb{P}(A_n)$.

Solution

a) Use $B = A \cup (B \backslash A)$, where A and $B \backslash A$ are disjoint, as well as $\Omega = A \cup A^c$ and $\mathbb{P}(\Omega) = 1$.

b) Use $A \cup B = A \cup (B \backslash (A \cap B))$ where A and $B \backslash (A \cap B)$ are disjoint, as well as a).

c) Use $\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} B_n$, where $B_n = A_n \backslash (A_1 \cup \ldots \cup A_{n-1})$; the B_n are disjoint, so by axiom (ii) and a),

$$\mathbb{P}(\cup_{n=1}^{\infty} A_n) = \mathbb{P}(\cup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mathbb{P}(B_n) \leq \sum_{n=1}^{\infty} \mathbb{P}(A_n)$$

d) $\mathbb{P}(\cup_{n\geq 1} A_n) = \mathbb{P}(\cup_{n\geq 1}(A_n \cap A_{n-1}^c)) \overset{(*)}{=} \sum_{n=1}^{\infty} \mathbb{P}(A_n \cap A_{n-1}^c) = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{P}(A_i \cap A_{i-1}^c)$ $\overset{(**)}{=} \lim_{n\to\infty} \mathbb{P}(\cup_{i=1}^{n}(A_i \cap A_{i-1}^c)) = \lim_{n\to\infty} \mathbb{P}(\cup_{i=1}^{n} A_i) = \lim_{n\to\infty} \mathbb{P}(A_n)$, where $(*)$, $(**)$ follow from the fact that the sets $A_n \cap A_{n-1}^c$ are disjoint.

e) Using parts a) and d): $\mathbb{P}(\cap_{n\geq 1} A_n) = 1 - \mathbb{P}((\cap_{n\geq 1} A_n)^c) = 1 - \mathbb{P}(\cup_{n\geq 1} A_n^c) = 1 - \lim_{n\to\infty} \mathbb{P}(A_n^c) = \lim_{n\to\infty} \mathbb{P}(A_n)$.

Problem 2: CDF

a) Which of the following are cdfs?

1. $F_1(t) = \exp(-e^{-t})$, $t \in \mathbb{R}$
2. $F_2(t) = \frac{1}{1-e^{-t}}$, $t \in \mathbb{R}$
3. $F_3(t) = 1 - \exp(-1/|t|)$, $t \in \mathbb{R}$
4. $F_4(t) = 1 - \exp(-e^{t})$, $t \in \mathbb{R}$

b) Let now F be a generic cdf.

Which of the following functions are guaranteed to be also cdfs?

5. $F_5(t) = F(t^2)$, $t \in \mathbb{R}$
6. $F_6(t) = F(t)^2$, $t \in \mathbb{R}$
7. $F_7(t) = F(1 - \exp(-t))$, $t \in \mathbb{R}$
8. $F_8(t) = \begin{cases} 1 - \exp(-F(t)/(1-F(t))) & \text{if } F(t) < 1 \\ 1 & \text{if } F(t) = 1 \end{cases} \quad t \in \mathbb{R}$

Solution a) 1. true, 2. false, 3. false, 4. true b) 5. false, 6. true, 7. false, 8. true.

Problem 3: Jumps of the CDF

Let X be an arbitrary random variable and $\mathbb{F}_X$ its CDF. Show that $\mathbb{F}_X$ can have at most a discrete number of jumps.

Solution

Let $D \in \mathbb{R}$ be the set of the jumps in $\mathbb{F}_X$. Assume that D is not finite. We show that D is countable by enumerating all the elements in it using the following procedure. At time $n = 1$, if there is an elements $x \in D$ such that $\mathbb{P}(\{x\}) \geq \frac{1}{2}$, we assign one to this element. Now, suppose we have enumerated i elements in the first $n-1$ steps of this procedure. At time n we consider all $x \in D$ such that $\mathbb{P}(\{x\}) \geq \frac{1}{n+1}$ and enumerate them as $i+1, i+2, \ldots$.

It is clear that at any time n we will enumerate at most n elements of D. Moreover, for each element of D there is N sufficiently large that this element will be enumerated at time N. Thus, the set D is at most countably infinite.

Problem 4: Sigma field generated by random variables

a) Let X, Y be two random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G} = \sigma(X) \cap \sigma(Y)$ [fact: it can be shown that $\mathcal{G}$ is a σ-field]. Is it true that $\{X \leq Y\} \in \mathcal{G}$?

b) Let X, Y be two independent random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Is it always true that $\sigma(X+Y) = \sigma(X, Y)$?

c) Let X be a continuous random variable whose pdf p_X is a continuous function on $\mathbb{R}$.
Let now $Y = X^2$. Is it always true that the pdf p_Y is also a continuous function on $\mathbb{R}$?

d) Let F be a generic cdf. Is it always true that the function $G : \mathbb{R} \to [0,1]$ defined as

$$G(t) = F(t^3 + 3t^2 + 3t + 1), \quad t \in \mathbb{R}$$

is also a cdf?

Solution a) No. Take for example $\Omega = \{1,2,3\}$, $X(\omega) = \omega$ and $Y(\omega) = 2$ for every $\omega \in \Omega$. Then $\mathcal{G} = \sigma(X) \cap \sigma(Y) = \sigma(Y) = \{\emptyset, \Omega\}$, but $\{X \leq Y\} = \{\omega \in \Omega : X(\omega) \leq Y(\omega)\} = \{1,2\} \notin \mathcal{G}$.

b) No. Take for example $\Omega = \{1,2\}^2$, $X(\omega) = \omega_1$ and $Y(\omega) = -\omega_2$. Then $\{X + Y = 0\} = \{(1,1),(2,2)\}$, and so $\sigma(X+Y) = \sigma(\{(1,1),(2,2)\}, \{(1,2)\}, \{(2,1)\}) \neq \sigma(X,Y) = \mathcal{P}(\Omega)$ (in addition, note that the fact that X and Y are independent does not play a role here).

c) No. Take for example $X \sim \mathcal{N}(0,1)$, whose pdf $p_X(x) = \frac{1}{\sqrt{2\pi x}} \exp(-x^2/2)$ is continuous. Then $Y = X^2$ has pdf

$$p_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi y}} \exp(-y/2) & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$$

which is discontinuous in $y = 0$.

d) Yes. Actually, the map $t \mapsto t^3 + 3t^2 + 3t + 1 = (t+1)^3$ is non-decreasing and going from $-\infty$ to $+\infty$, thus the properties of the cdf F are preserved for G.