Introduction to Quantum Information Processing
COM 309 Week 1

Complex Numbers

Exercise 1

1. Give the real part, imaginary part, and conjugate of the following complex numbers:

$$z_1 = 3 + 2i \qquad z_2 = i(1 + \sqrt{2}i) \qquad z_3 = \frac{5}{i} \qquad z_4 = \frac{i^4}{2} + 2i^3 - i^2 + i - 3$$

2. Give the modulus and argument of the following complex numbers:

$$z_1 = 2i \qquad z_2 = 1 + \sqrt{3}i \qquad z_3 = -5 \qquad z_4 = a + ib$$

3. Give the algebraic form (cartesian coordinates) of the following complex numbers:

$$z_1 = (2 - 3i)(1 + \sqrt{2}i) \qquad z_2 = (\overline{1 + 3i})(6 - \sqrt{3}i) \qquad z_3 = \frac{5 + i}{3 - i} \qquad z_4 = (1 - i)^5$$

4. Let $z_1 = 3 + 2i$, $z_2 = 1 - i$. Plot in the complex plane $z_1, z_2, z_3 = z_1 + z_2, z_4 = z_1 - \bar{z}_2$, $z_5 = \bar{z}_1 z_2$.

Exercise 2

1. Give the polar form (or exponential form) of the following complex numbers:

$$z_1 = 1 + e^{i\theta} \qquad z_2 = 1 - e^{-i\theta} \qquad z_3 = e^{i\theta} + e^{i\phi} \qquad z_4 = \frac{1 + e^{i\theta}}{1 + e^{i\phi}}$$

2. Solve the equation $z^n = 1$ and plot the solutions for $n = 5$ in the complex plane.

3. Solve the equation $z^2 = \frac{1+i}{\sqrt{2}}$ and give the solutions in algebraic form. Deduce the value of $\cos\left(\frac{\pi}{8}\right)$ and $\sin\left(\frac{\pi}{8}\right)$.

Exercise 3

Find the scalar product $\langle \mathbf{v}, \mathbf{w} \rangle$ of the following pairs of vectors in $\mathbb{C}^d$:

1. $d = 2$, $\mathbf{v} = (1+i, 2+3i)$, $\mathbf{w} = (4-2i, 3+i)$;
2. $d = 2$, $\mathbf{v} = (4-2i, 3+i)$, $\mathbf{w} = (1+i, 2+3i)$;
3. $d = 3$, $\mathbf{v} = (2-i, 1, -3i)$, $\mathbf{w} = (4, -i, 2+i)$;
4. $d = 3$, $\mathbf{v} = (1+2i, 2-i, 3+i)$, $\mathbf{w} = (1, 2i, 3-i)$.

Exercise 4

Check whether the following vectors in $\mathbb{C}^d$ form an orthonormal basis :

1. $d = 2$, $\mathbf{u}_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} \frac{1}{2} \\ -\frac{i}{2} \end{pmatrix}$;
2. $d = 2$, $\mathbf{u}_1 = \frac{1}{2} \begin{pmatrix} 1+i \\ 1-i \end{pmatrix}$, $\mathbf{u}_2 = \frac{1}{2} \begin{pmatrix} 1-i \\ 1+i \end{pmatrix}$;
3. $d = 3$, $\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{u}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ i \\ 1 \end{pmatrix}$, $\mathbf{u}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}$;
4. $d = 3$, $\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$, $\mathbf{u}_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1+i \\ 1-i \\ 1-i \end{pmatrix}$, $\mathbf{u}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -i \\ 2i \end{pmatrix}$.

Exercise 5

1. Prove that all eigenvalues $\lambda_1, \ldots, \lambda_d$ of arbitrary Hermitian matrix $H \in \mathbb{C}^{n \times n}$ are real.
2. Prove also that the eigenvectors of distinct eigenvalues are orthogonal.
3. Deduce that we can always choose the set of eignevectors as an orthonormal basis $v_1, \ldots, v_d$.
4. Finally deduce that for a Hermitian matrix we have the decomposition

$$H = \sum_{i=1}^{d} \lambda_i v_i v_i^\dagger$$

Exercise 6

Show that the matrix $A = \begin{pmatrix} 3 & 1-i \\ 1+i & 4 \end{pmatrix} \in \mathbb{C}^{2 \times 2}$ is Hermitian and find its eigenvalues and eigenvectors.

Exercise 7

Let U be an arbitrary unitary matrix in $\mathbb{C}^{n\times n}$.

1. Show that a unitary matrix preserves the scalar product: $(U\mathbf{w})^\dagger U\mathbf{z} = \mathbf{w}^\dagger \cdot \mathbf{z}$.
2. Show that a unitary matrix preserves the norm of a vector $\mathbf{z} \in \mathbb{C}^n$, $\|U\mathbf{z}\| = \|\mathbf{z}\|$.
3. Show that the lines of a unitary matrix form an orthonormal basis. Show that the columns also form an orthonormal basis.
4. Show that the modulus of any eigenvalue of U is 1.

Exercise 8

Show that matrix

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \theta \in \mathbb{R}$$

is unitary and find its eigenvalues and eigenvectors.

Solutions

Exercise 1

1. Give the real part, imaginary part, and conjugate of the following complex numbers:

$$z_1 = 3 + 2i \qquad z_2 = i(1 + \sqrt{2}i) \qquad z_3 = \frac{5}{i} \qquad z_4 = \frac{i^4}{2} + 2i^3 - i^2 + i - 3$$

   (a) $\mathrm{Re}(z_1) = 3$, $\mathrm{Im}(z_1) = 2$, $\overline{z_1} = 3 - 2i$
   (b) $\mathrm{Re}(z_2) = -\sqrt{2}$, $\mathrm{Im}(z_2) = 1$, $\overline{z_2} = 1 + \sqrt{2}i$
   (c) $\mathrm{Re}(z_3) = 0$, $\mathrm{Im}(z_3) =$ -5, $\overline{z_3} = 5i$
   (d) $\mathrm{Re}(z_4) = -\frac{3}{2}$, $\mathrm{Im}(z_4) =$ -1, $\overline{z_4} = -\frac{3}{2} + i$

2. Give the modulus and argument of the following complex numbers:

$$z_1 = 2i \qquad z_2 = 1 + \sqrt{3}i \qquad z_3 = -5 \qquad z_4 = a + ib$$

   (a) $|z_1| = 2$, $\arg(z_1) = \frac{\pi}{2}$
   (b) $|z_2| = 2$, $\arg(z_2) = \frac{\pi}{3}$
   (c) $|z_3| = 5$, $\arg(z_3) = -\pi$
   (d) $|z_4| = \sqrt{a^2 + b^2}$, $\arg(z_4) = \tan^{-1}\left(\frac{b}{a}\right)$

3. Give the algebraic form of the following complex numbers:

$$z_1 = (2 - 3i)(1 + \sqrt{2}i) \qquad z_2 = (\overline{1 + 3i})(6 - \sqrt{3}i) \qquad z_3 = \frac{5 + i}{3 - i} \qquad z_4 = (1 - i)^5$$

   (a) $z_1 = (2 + 3\sqrt{2}) + i(2\sqrt{2} - 3)$
   (b) $z_2 = (6 - 3\sqrt{3}) + i(\sqrt{3} - 18)$
   (c) $z_3 = \frac{1}{10}(14 + 8i)$
   (d) $z_4 = \sqrt{2}^5 e^{-i\frac{5\pi}{4}} = -4 + 4i$

4. Let $z_1 = 3 + 2i$, $z_2 = 1 - i$. Plot in the complex plane $z_1, z_2, z_3 = z_1 + z_2, z_4 = z_1 - \bar{z}_2$, $z_5 = \bar{z}_1 z_2$.

   See Fig. 1.

Exercise 2

1. Give the exponential form of the following complex numbers:

$$z_1 = 1 + e^{i\theta} \qquad z_2 = 1 - e^{-i\theta} \qquad z_3 = e^{i\theta} + e^{i\phi} \qquad z_4 = \frac{1 + e^{i\theta}}{1 + e^{i\phi}}$$

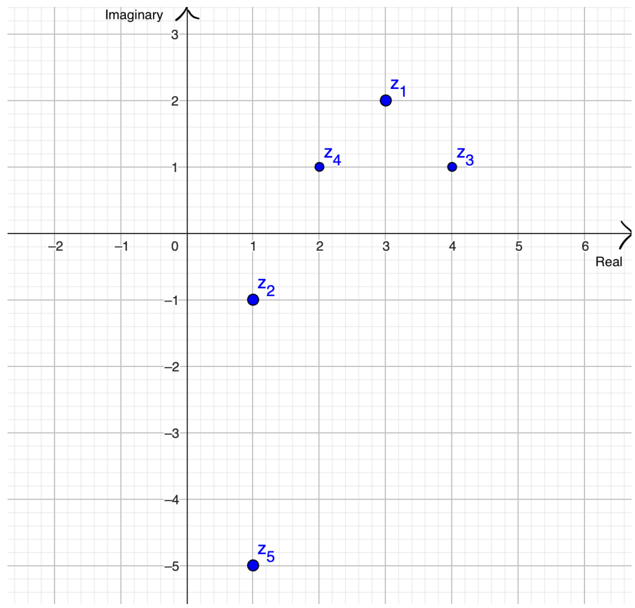

Figure 1: Ex 1.4

(a) $z_1 = 2\cos(\frac{\theta}{2})e^{i\frac{\theta}{2}}$

(b) $z_2 = 2\sin(\frac{\theta}{2})e^{i\frac{-\theta+\pi}{2}}$

(c) $z_3 = 2\cos(\frac{\theta-\phi}{2})e^{i\frac{\theta+\phi}{2}}$

(d) $z_4 = \frac{\cos(\frac{\theta}{2})}{\cos(\frac{\phi}{2})}e^{i\frac{\theta-\phi}{2}}$

2. Solve the equation $z^n = 1$ and plot the solutions for $n = 5$ in the complex plane.
$z = e^{i\frac{2k\pi}{n}}$ for $k = [\![0, \dots, n-1]\!]$. See Fig. 2 for the plot.

3. Solve the equation $z^2 = \frac{1+i}{\sqrt{2}}$ in algebraic form. Deduce the value of $\cos\left(\frac{\pi}{8}\right)$ and $\sin\left(\frac{\pi}{8}\right)$.
Let's write $z = a + ib$, then $\frac{1+i}{\sqrt{2}} = (a+ib)^2 = a^2 - b^2 + 2iab$. So $a^2 - b^2 = \frac{\sqrt{2}}{2}$ and $2ab = \frac{\sqrt{2}}{2}$. Solving this system gives $a^2 = \frac{2+\sqrt{2}}{4}$ and $b^2 = \frac{2-\sqrt{2}}{4}$ and thus $a = \pm\sqrt{\frac{2+\sqrt{2}}{2}}$ and $b = \pm\frac{\sqrt{2-\sqrt{2}}}{2}$. The second equality $2ab = \frac{\sqrt{2}}{2}$ tells us that ab have the same sign so the two solutions are $z_1 = \frac{\sqrt{2+\sqrt{2}}}{2} + i\frac{\sqrt{2-\sqrt{2}}}{2}$ and $z_2 = -z_1$.
Let's note that $z^2 = \frac{1+i}{\sqrt{2}} = e^{\frac{i\pi}{4}}$ so the two solutions in exponential form are $z_1 = e^{\frac{i\pi}{8}}$ and $z_2 = -e^{\frac{i\pi}{8}} = e^{\frac{9i\pi}{8}}$. We can identify the real part and the imaginary part to obtain $\cos\left(\frac{\pi}{8}\right) = \frac{\sqrt{2+\sqrt{2}}}{2}$ and $\sin\left(\frac{\pi}{8}\right) = \frac{\sqrt{2-\sqrt{2}}}{2}$.

Exercise 3

Find the scalar product $\langle \mathbf{v}, \mathbf{w} \rangle$ of the following pairs of vectors in $\mathbf{V}$:

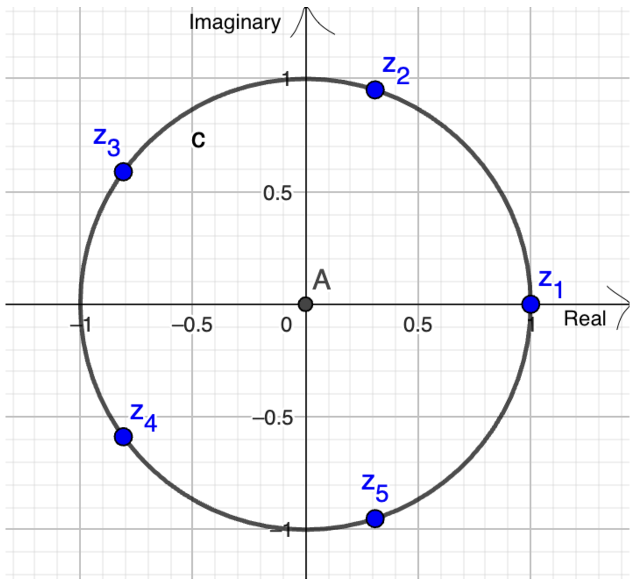

Figure 2: Ex 2.2

1. $\mathbf{V} = \mathbb{C}^2$, $\mathbf{v} = (1+i, 2+3i)$, $\mathbf{w} = (4-2i, 3+i)$;
2. $\mathbf{V} = \mathbb{C}^2$, $\mathbf{v} = (4-2i, 3+i)$, $\mathbf{w} = (1+i, 2+3i)$;
3. $\mathbf{V} = \mathbb{C}^3$, $\mathbf{v} = (2-i, 1, -3i)$, $\mathbf{w} = (4, -i, 2+i)$;
4. $\mathbf{V} = \mathbb{C}^3$, $\mathbf{v} = (1+2i, 2-i, 3+i)$, $\mathbf{w} = (1, 2i, 3-i)$.

Answers: 1. $11-13i$ 2. $11+13i$ 3. $5+9i$ 4. $7-4i$.

Exercise 4

Check whether the following vectors in $\mathbb{C}^d$ form an orthonormal basis :

1. $d = 2$, $\mathbf{u}_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} \frac{1}{2} \\ -\frac{i}{2} \end{pmatrix}$;
2. $d = 2$, $\mathbf{u}_1 = \frac{1}{2}\begin{pmatrix} 1+i \\ 1-i \end{pmatrix}$, $\mathbf{u}_2 = \frac{1}{2}\begin{pmatrix} 1-i \\ 1+i \end{pmatrix}$;
3. $d = 3$, $\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{u}_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ i \\ 1 \end{pmatrix}$, $\mathbf{u}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}$;
4. $d = 3$, $\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$, $\mathbf{u}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1+i \\ 1-i \\ 1-i \end{pmatrix}$, $\mathbf{u}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -i \\ 2i \end{pmatrix}$.

Answers: 1. no, as $\|\mathbf{u}_1\| = \|\mathbf{u}_1\| = \frac{1}{2} \neq 0$; 2. yes;
3. no, because $\langle \mathbf{u}_1, \mathbf{u}_3 \rangle = -\frac{i}{2} \neq 0$; 4. yes.

Exercise 5

1. Prove that all eigenvalues of arbitrary Hermitian matrix $H \in \mathbb{C}^{n \times n}$ are real.

   Solution: Let $\lambda \in \mathbb{C}$ and $\mathbf{z} \in \mathbb{C}^n$ be an eigenvalue of H and corresponding eigenvector, i.e. $H\mathbf{z} = \lambda \mathbf{z}$. Then

   $$\lambda \mathbf{z}^\dagger \mathbf{z} = \mathbf{z}^\dagger(\lambda \mathbf{z}) = \mathbf{z}^\dagger(H\mathbf{z}) = \mathbf{z}^\dagger H^\dagger \mathbf{z} = (H\mathbf{z})^\dagger \mathbf{z} = (\lambda \mathbf{z})^\dagger \mathbf{z} = \overline{\lambda} \mathbf{z}^\dagger \mathbf{z}.$$

   As $\mathbf{z}^\dagger \mathbf{z} = \|\mathbf{z}\|^2 \neq 0$, $\lambda = \overline{\lambda}$ and therefore λ is real.

2. Prove also that the eigenvectors of distinct eigenvalues are orthogonal.

   Solution: Consider teh eigenvalue equation for two distinct eigenvalues $\lambda_i \neq \lambda_j$. We have $Hv_i = \lambda_i v_i$ and $Hv_j = \lambda_j v_j$. The second equation can be conjugated and since $H = H^\dagger$ we have $v_j^\dagger H = \lambda_j v_j^\dagger$. Multiplying the first equation by $v_j^\dagger$ we get $v_j^\dagger H v_i = \lambda_i v_j^\dagger v_i$ and thus $\lambda_j v_j^\dagger v_i = \lambda_i v_j^\dagger v_i$ and since eigenvalues are distinct we obtain $v_j^\dagger v_i = 0$.

3. Deduce that we can always choose the set of eigenvectors as an orthonormal basis $v_1, \ldots, v_d$.

   Solution: There are d eigenvalues and d eigenvectors. We can normlize all eigenvectors so they get norm 1. For distinct eigenvalues the eigenvectors are orthogonal, and for equal eigenvalues we can choose them orthogonal by apropriate rotation (in the eigensubspace). Therefore for an Hermitian matrix we can always take a set of eigenvectors that form an orthonormal basis. *Remark*: eigenvectors that correspond to non-degenerate eigenvalues are unique up to normalization.

4. Finally deduce that for a Hermitian matrix we have the decomposition

   $$H = \sum_{i=1}^{d} \lambda_i v_i v_i^\dagger$$

   Solution: From $Hv_i = \lambda_i v_i$ we deduce

   $$\sum_{i=1}^{d} (Hv_i) v_i^\dagger = \sum_{i=1}^{d} \lambda_i v_i v_i^\dagger$$

Thus

$$H \sum_{i=1}^{d} v_i v_i^\dagger = \sum_{i=1}^{d} \lambda_i v_i v_i^\dagger$$

Now since we can choose the eigenvectors to form an orthonormal basis we have that

$$\sum_{i=1}^{d} v_i v_i^\dagger = I$$

and the claimed result follows.

Remark: to see this last point it suffices to note that for any vector w:

$$(\sum_{i=1}^{d} v_i v_i^\dagger) w = \sum_{i=1}^{d} v_i (v_i^\dagger w) = w = Iw$$

Exercise 6

Show that the matrix $A = \begin{pmatrix} 3 & 1-i \\ 1+i & 4 \end{pmatrix} \in \mathbb{C}^{2\times 2}$ is Hermitian and find its eigenvalues and eigenvectors.

Solution: We find characteristic polynomial $\det(A - \lambda \mathbb{I}) = 10 - 7\lambda + \lambda^2$; its solutions are $\lambda_1 = 2$ and $\lambda_2 = 5$. Then, by finding vectors satisfying $A - \lambda \mathbb{I} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, we see that $\mathbf{v}_1 = \begin{pmatrix} 2 \\ -1-i \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1-i \\ 2 \end{pmatrix}$ (up to scaling) are corresponding eigenvectors.

Exercise 7

Let U be an arbitrary unitary matrix in $\mathbb{C}^{n\times n}$.

1. Show that a unitary matrix preserves the scalar product: $(U\mathbf{w})^\dagger U\mathbf{z} = \mathbf{w}^\dagger \cdot \mathbf{z}$.

   Solution:

   $$(U\mathbf{w})^\dagger (U\mathbf{z}) = \mathbf{w}^\dagger U^\dagger U \mathbf{z} = \mathbf{w}^\dagger \mathbf{z}$$

   where we used unitarity $U^\dagger U = I$

2. Show that for a vector $\mathbf{z} \in \mathbb{C}^n$, $\|U\mathbf{z}\| = \|\mathbf{z}\|$.

   Solution: Take $\mathbf{w} = \mathbf{z}$ in the little proof above. Or just repeat it:

   $$\|U\mathbf{z}\|^2 = (U\mathbf{z})^\dagger (U\mathbf{z}) = \mathbf{z}^\dagger U^\dagger U \mathbf{z} = \mathbf{z}^\dagger \mathbf{z} = \|\mathbf{z}\|^2 \Rightarrow \|U\mathbf{z}\| = \|\mathbf{z}\|$$

3. Show the lines and columns are orthonormal.

   Solution: This follows by writing down in matrix element form the conditions of unitarity: $U^\dagger U = I$ and $UU^\dagger = I$. For example the first condition is

$$\sum_{k=1}^{d} U^\dagger_{ik} U_{kj} = \delta_{ij}$$

   which is equivalent to

$$\sum_{k=1}^{d} U^*_{ki} U_{kj} = \delta_{ij}$$

   For $i = j$ this means column i is a vector of norm 1, and for $i \neq j$ the two columns i and j are orthogonal vectors. There are d columns and thus they form an orthonormal basis. The same arguments with the other unitarity condition shows that the same holds for the lines.

4. Show that the modulus of any eigenvalue of U is 1.

   Solution: Let $\lambda \in \mathbb{C}$ and $\mathbf{v} \in \mathbb{C}^n$ be eigenvalue and corresponding eigenvector of U. Then,

$$U\mathbf{v} = \lambda\mathbf{v} \to \|U\mathbf{v}\| = \|\lambda\mathbf{v}\|.$$

   On the one hand, from the previous result, we have $\|U\mathbf{v}\| = \|\mathbf{v}\|$; on the other, $\|\lambda\mathbf{v}\| = |\lambda|\|\mathbf{v}\|$. From this, we get $|\lambda| = 1$.

Exercise 8

Show that matrix

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \theta \in \mathbb{R}$$

is unitary and find its eigenvalues and eigenvectors.

Solution:

$$U^\dagger U = U^\top U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} =$$
$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which proves that U is unitary. The characteristic polynomial is:

$$\det(U - \lambda\mathbb{I}) = (\lambda - \cos\theta)^2 + \sin^2\theta = (\lambda - \cos\theta - i\sin\theta)(\lambda - \cos\theta + i\sin\theta),$$

which means that $\lambda_{1,2} = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$. The eigenvectors then are $\mathbf{v}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$ for $\lambda_1 = e^{i\theta}$ and $\mathbf{v}_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$ for $\lambda_2 = e^{-i\theta}$ (up to scaling).