Math Recap.

Here we review a few basic notions of linear algebra used all along and notably next week to formulate the axioms. In particular we introduce the Dirac notation for vectors, matrices, ...

The arena of quantum mechanics is the complex Hilbert space. This is just a vector space $\mathcal{H}$ over the field of complex numbers $\mathbb{C}$. In quantum information & computation, at least in this class, we only need finite dimensional vector spaces, say $\dim \mathcal{H} = d$.

- Vectors $\vec{\psi}$ are column vectors $\begin{pmatrix} \psi_1 \\ \vdots \\ \psi_d \end{pmatrix}$, $\psi_i \in \mathbb{C}$

  They are denoted $|\psi\rangle$ (instead of $\vec{\psi}$). The ket.

- Conjugate vectors are line vectors. By definition

  $$\vec{\psi}^{T,*} \equiv \vec{\psi}^{+} = (\psi_1^*, \dots, \psi_d^*)$$

  They are denoted $\langle\psi|$. the bra

  So $(|\psi\rangle)^{T,*} \equiv |\psi\rangle^{+} = \langle\psi|$.

- In the Hilbert space there is a scalar product:

  $$\underbrace{\vec{\psi}^{+} \cdot \vec{\varphi}}_{\in \mathbb{C}} = \sum_{i=1}^{d} \psi_i^* \varphi_i \quad \text{denoted } \langle\psi|\varphi\rangle$$

  the bracket

Properties of scalar (or inner) product.

- skew symmetry $\langle \psi | \varphi \rangle^* = \langle \varphi | \psi \rangle$.
- bilinear $\langle \psi | \left(\alpha |\varphi_1\rangle + \beta |\varphi_2\rangle \right) = \alpha \langle \psi | \varphi_1 \rangle + \beta \langle \psi | \varphi_2 \rangle$

  Note by (Dirac) conjugation:

$$\left(\bar{\alpha} \langle \varphi_1 | + \bar{\beta} \langle \varphi_2 | \right) |\psi\rangle = \bar{\alpha} \langle \varphi_1 | \psi \rangle + \bar{\beta} \langle \varphi_2 | \psi \rangle$$

- Positive definite $\underbrace{\langle \psi | \psi \rangle}_{\in \mathbb{R}_+} \geq 0$ and equality iff $|\psi\rangle = 0$.

Other properties.

- $\sqrt{\langle \psi | \psi \rangle} = \| \psi \|$ defines a norm.

  So $\| \psi \|^2 = \langle \psi | \psi \rangle$.

- triangle inequality $\| \psi + \varphi \| \leq \| \psi \| + \| \varphi \|$
- Cauchy-Schwarz inequality $| \langle \psi | \varphi \rangle | \leq \| \psi \| \| \varphi \|$

Orthonormal basis

If $\dim \mathcal{H} = d$, this is a set of d vectors $|\varphi_1\rangle \ldots |\varphi_d\rangle$

such that $\langle \varphi_i | \varphi_j \rangle = \delta_{ij}$. That is

$$\begin{cases} \langle \varphi_i | \varphi_i \rangle = 1 & (\text{for } i = j) \\ \langle \varphi_i | \varphi_j \rangle = 0 & \text{if } i \neq j \end{cases}$$

Example of Hilbert spaces

- $\mathbb{C}^2$: $\alpha|0\rangle + \beta|1\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$, $\alpha, \beta \in \mathbb{C}$. kets

  $\underbrace{|0\rangle}_{\begin{pmatrix}1\\0\end{pmatrix}}$ $\underbrace{|1\rangle}_{\begin{pmatrix}0\\1\end{pmatrix}}$

  $\bar{\alpha}\langle 0| + \bar{\beta}\langle 1| = (\bar{\alpha}, \bar{\beta})$ bras

  This is the Hilbert space of a qubit.

- $\mathbb{C}^d$: $\sum_{i=1}^{d} \alpha_i \underbrace{|i\rangle}_{} = \begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_d\end{pmatrix}$ $\alpha_i \in \mathbb{C}$ kets

  canonical basis next say.

  $\sum_{i=1}^{d} \alpha_i^* \langle i|$ bras

  This is the Hilbert space of a qudit.

- $L^2(\mathbb{R}^3)$ space of functions $\underline{x} \in \mathbb{R}^3 \longmapsto \psi(\underline{x}) \in \mathbb{C}$.

  such that $\int |\psi(\underline{x})|^2 \, d\underline{x} < \infty$.

  In this class we do not use this space.

  Inner product is $\langle \psi | \varphi \rangle = \int_{\mathbb{R}^3} \psi^*(\underline{x}) \, \varphi(\underline{x}) \, d\underline{x}$

  Another notation for $\psi(\underline{x})$ is $\langle \underline{x} | \psi \rangle$.

Tensor Product.

From two (or more) Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ we can form a bigger space $\mathcal{H}_A \otimes \mathcal{H}_B$ "$\equiv \mathcal{H}_{A \cup B}$".

A concrete construction is as follows:

- $\mathcal{H}_A$ has an orthonormal basis $|v_1\rangle \dots |v_{d_A}\rangle$, $\dim \mathcal{H}_A = d_A$
- $\mathcal{H}_B$ has an " " $|w_1\rangle \dots |w_{d_B}\rangle$, $\dim \mathcal{H}_B = d_B$.

- $\mathcal{H}_A \otimes \mathcal{H}_B$ is the vector span with orthonormal basis given by all pairings

$$\{ |v_i\rangle \otimes |w_j\rangle \qquad i = 1 \dots d_A \,;\, j = 1 \dots d_B \}$$

so there are $d_A d_B = \dim(\mathcal{H}_A \otimes \mathcal{H}_B)$ $\overbrace{\text{vectors}}^{\text{basis}}$.

A general vector in $\mathcal{H}_A \otimes \mathcal{H}_B$ is

$$|\psi\rangle = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} \alpha_{ij} \, |v_i\rangle \otimes |w_j\rangle$$

$$= \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} \alpha_{ij} \, |v_i, w_j\rangle \qquad \text{(other notation)}.$$

$d_A d_B$ components.

- Inner product given by

$$(\langle \varphi_A | \otimes \langle \varphi_B |)(|\psi_A\rangle \otimes |\psi_B\rangle)$$

$$= \underbrace{\langle \varphi_A | \psi_A \rangle}_{\in \mathbb{C}} \underbrace{\langle \varphi_B | \psi_B \rangle}_{\in \mathbb{C}}.$$

Thus it is easy to see that $\{|v_i\rangle \otimes |w_j\rangle\}$ is orthonormal if $\{|v_i\rangle\}$ and $\{|w_j\rangle\}$ are orthonormal.

- Last property: product $\otimes$ is distributive over addition + (hence $\otimes$ notation). This means

$$|\varphi\rangle \otimes (\alpha|\psi_1\rangle + \beta|\psi_2\rangle) = \alpha|\varphi\rangle\otimes|\psi_1\rangle + \beta|\varphi\rangle\otimes|\psi_2\rangle$$

- Conjugation acts like

$$(|\varphi\rangle \otimes |\psi\rangle)^{+} = \langle\varphi| \otimes \langle\psi| .$$

– With all these rule the tensor product is fully defined. –

Examples.

- $\mathbb{C}^2 \otimes \mathbb{C}^2$ is the space of vectors of type spanned ~~xxxxx~~ by the orthonormal basis

$$|0\rangle\otimes|0\rangle \;,\; |0\rangle\otimes|1\rangle \;,\; |1\rangle\otimes|0\rangle \;,\; |1\rangle\otimes|1\rangle$$

So general elements of $\mathbb{C}^2\otimes\mathbb{C}^2$ are

$$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle .$$

$$= \begin{pmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{pmatrix}$$ 4 dim vector assuming $|00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}; |01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$

$|10\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}; |11\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

- In particular we have elements of the type (product state)

$$(\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$$

$$= \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle .$$

- In coordinates this is

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \otimes \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix}$$

This is a "rule" for computing tensor products in coordinates.

- Conjugate vects

$$\langle\psi| = \overline{a_{00}}\langle 00| + \overline{a_{01}}\langle 01| + \overline{a_{10}}\langle 10| + \overline{a_{11}}\langle 11|$$

$$= (\overline{a_{00}}, \overline{a_{01}}, \overline{a_{10}}, \overline{a_{11}})$$

$$(\bar{\alpha}\langle 0| + \bar{\beta}\langle 1|) \otimes (\bar{\gamma}\langle 0| + \bar{\delta}\langle 1|)$$

$$= \bar{\alpha}\bar{\gamma}\langle 00| + \bar{\alpha}\bar{\delta}\langle 01| + \bar{\beta}\bar{\gamma}\langle 10| + \bar{\beta}\bar{\delta}\langle 11|$$

This reads in coordinates

$$(\bar{\alpha}, \bar{\beta}) \otimes (\bar{\gamma}, \bar{\delta}) = (\bar{\alpha}\bar{\gamma}, \bar{\alpha}\bar{\delta}, \bar{\beta}\bar{\gamma}, \bar{\beta}\bar{\delta}).$$

Matrices

- Let $\mathcal{H}$ have dimension d.

square $d \times d$ matrix $A : \mathcal{H} \to \mathcal{H}$

$$|\psi\rangle \mapsto A|\psi\rangle$$

is just a linear map satisfying $A(|\varphi\rangle + |\psi\rangle) = A|\varphi\rangle + A|\psi\rangle$.

- The matrix can be represented in a basis $\{|1\rangle \ldots |d\rangle\}$

by a tableau of matrix elements $A_{ij} = \underbrace{\underbrace{\langle i|}_{\text{line}} A \underbrace{|j\rangle}_{\text{column}}}_{\text{bracket.}} \in \mathbb{C}$.

Property:

$$A = \sum_{i,j=1}^{d} A_{ij} |i\rangle\langle j| \,. \quad \text{for (ortho) basis.}$$

Proof:

$$\sum_{i,j} A_{ij} |i\rangle\langle j|$$

$$= \sum_{i,j} \langle i|A|j\rangle |i\rangle\langle j|$$

$$= \sum_{i,j} |i\rangle \langle i|A|j\rangle\langle j|$$

$$= \underbrace{\left(\sum_i |i\rangle\langle i|\right)}_{\mathbb{1}} A \underbrace{\left(\sum_j |j\rangle\langle j|\right)}_{\mathbb{1}} = A$$

identity matrix.

Note $\left(\sum_i |i\rangle\langle i|\right)|v\rangle = \sum_i |i\rangle \underbrace{\langle i|v\rangle}_{\substack{\text{projection of } \vec{v} \\ \text{on } \vec{i}^T}}$.

$= \text{decomposition of } \vec{v} \text{ in basis } |1\rangle \ldots |d\rangle$

$= \vec{v} = |v\rangle$.

Thus indeed if $|1\rangle \ldots |d\rangle$ then $\sum_i |i\rangle\langle i| = \mathbb{1}$.

- Adjoint of a matrix.

By definition the adjoint of A is the matrix A^+ which satisfies $\langle v|A^+|w\rangle = \overline{\langle w|A|v\rangle}$.

In particular if $|1\rangle \ldots |d\rangle$ is a basis we have

$$\langle i|A^+|j\rangle = \overline{\langle j|A|i\rangle}$$

$$(A^+)_{ij} = A^*_{ji}$$

We see that the adjoint = transpose & complex conjugate

$$A^+ = A^{T,*}.$$

- If $A = \sum_{i,j} A_{ij} |i\rangle\langle j|$

Then $A^+ = \sum_{i,j} A^*_{ji} |j\rangle\langle i|$.

Tensor product of matrices.

$$A : \mathcal{H}_1 \to \mathcal{H}_1 \qquad d_1 \times d_1$$
$$B : \mathcal{H}_2 \to \mathcal{H}_2 \qquad d_2 \times d_2 .$$

$$A \otimes B : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$$

$$|\psi_1\rangle \otimes |\psi_2\rangle \to (A \otimes B)|\psi_1\rangle \otimes |\psi_2\rangle \equiv A|\psi_1\rangle \otimes B|\psi_2\rangle$$

+ linearity of $A \otimes B$. $\boxed{\text{Note } A \otimes B \text{ is } d_1 d_2 \times d_1 d_2}$

Tableau representations:

$$A \otimes B = \left(\sum_{ij} A_{ij} |i\rangle\langle j| \right) \otimes \left(\sum_{k,l} B_{kl} |k\rangle\langle l| \right)$$

$$= \sum_{ijkl} A_{ij} B_{kl} \, |i,k\rangle\langle j,l| .$$

So the matrix elements of $A \otimes B$ are

$$(A \otimes B)_{ik;jl} = A_{ij} B_{kl} .$$

One can see that this is equivalent to making

$$\underset{2\times 2}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \otimes \underset{2\times 2}{\begin{pmatrix} e & f \\ g & h \end{pmatrix}} = \underset{4\times 4}{\begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix}} .$$

Two types of matrices will play a particularly important role for us.

Hermitian matrices or **self-adjoint matrices**

A is an hermitian matrix (or self adjoint) if

$$A = A^{\dagger} \qquad \left(\begin{array}{ll} \text{i.e.} & A = A^{T,*} \\ \text{or} & A_{ij} = A_{ji}^{*} . \end{array} \right) .$$

Spectral theorem for hermitian matrix (very important) <u>exercises</u>

Let A be hermitian. Then all its eigenvalues <u>are real</u>. ~~The eigenvectors are orthogonal~~ One can choose all eigenvectors to be <u>orthonormal</u> and they form a basis of $\mathcal{H}$. In particular if

$$A |v_i\rangle = \underset{\substack{\text{eigenvalue} \\ \in \mathbb{R}}}{\lambda_i} \, \underset{\text{eigenvector.}}{|v_i\rangle}$$

we have

$$A = \sum_{i=1}^{d} \lambda_i \, |v_i\rangle\langle v_i|$$

$$\Big(\text{or } A = \sum_{i=1}^{d} \lambda_i \, \underbrace{\vec{v}_i \, \vec{v}_i^{\,T}}_{\text{projection matrix on } \vec{v}_i .} \; . \Big)$$

Unitary matrices.

U is unitary if $U^\dagger U = U U^\dagger = \mathbb{1}$.

Properties (exercises).

- $U^{-1} = U^\dagger$
- U preserves inner product. $(U|\psi\rangle)^\dagger (U|\varphi\rangle)$
  $= \langle\psi|\varphi\rangle$.
- U preserves norm $\|U\psi\| = \|\psi\|$.
- Lines of U are an orthonormal basis.
  Same for columns.

Examples.

The following are 2×2 hermitian matrices:

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$\sigma_z = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\{\sigma_x, \sigma_y, \sigma_z\}$ are called the Pauli matrices. We will encounter them often and also their properties.

Any 2×2 hermitian matrix can be writ $A = a_0 \mathbb{1} + a_1 \sigma_x + a_2 \sigma_y + a_3 \sigma_z$, and $a_0, a_1, a_2, a_3 \in \mathbb{R}$

Properties (exercises)

- $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{1}$.

$$\left\{ \begin{aligned} \sigma_x \sigma_y &= -\sigma_y \sigma_x \\ \sigma_x \sigma_z &= -\sigma_z \sigma_x \\ \sigma_x \sigma_z &= -\sigma_z \sigma_x \end{aligned} \right.$$

$$\left\{ \begin{aligned} [\sigma_x, \sigma_y] &= 2i \sigma_z \\ [\sigma_y, \sigma_z] &= 2i \sigma_x \\ [\sigma_z, \sigma_x] &= 2i \sigma_y \end{aligned} \right.$$

this is the Pauli matrix algebra.

Eigenvectors & eigenvalues of σ_z ; $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$

$+1$ $\qquad$ -1.

Eig vectors e eig val of σ_x : $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \rightarrow +1$

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \rightarrow -1$.

Eig vect & eig val of σ_y : do it yourself!

Stones theorem.

Let U be a unitary matrix. There exist an hermitian matrix H such that $U = \exp(iH)$.

Remark: The exponential of a matrix is by definition

$$\exp(M) = \sum_{k=0}^{+\infty} \frac{M^k}{k!} = \mathbb{1} + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \dots$$

Case of 2×2 unitary matrices. (will come back later in class).

They can always be written as

$$U = e^{iH} \quad \text{with} \quad H = a_0 \mathbb{1} + a_1 \sigma_x + a_2 \sigma_y + a_3 \sigma_z$$

and $a_0, a_1, a_2, a_3 \in \mathbb{R}$.

Eulers formula (follows from algebra of Pauli matrices)

$$e^{iH} = e^{ia_0} \cdot e^{i(a_1\sigma_x + a_2\sigma_y + a_3\sigma_z)}$$

$$= e^{ia_0} \left(\cos\alpha \, \mathbb{1} + i \sin\alpha \, \hat{n} \cdot \vec{\sigma} \right)$$

with $\alpha = \|\vec{a}\|$ and $\hat{n} = \frac{\vec{a}}{\|\vec{a}\|}$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$.