Introduction to Quantum Information Processing
COM 309 Week 1

Complex Numbers

Exercise 1

1. Give the real part, imaginary part, and conjugate of the following complex numbers:

$$z_1 = 3 + 2i \qquad z_2 = i(1+\sqrt{2}i) \qquad z_3 = \frac{5}{i} \qquad z_4 = \frac{i^4}{2} + 2i^3 - i^2 + i - 3$$

2. Give the modulus and argument of the following complex numbers:

$$z_1 = 2i \qquad z_2 = 1+\sqrt{3}i \qquad z_3 = -5 \qquad z_4 = a + ib$$

3. Give the algebraic form (cartesian coordinates) of the following complex numbers:

$$z_1 = (2-3i)(1+\sqrt{2}i) \qquad z_2 = (\overline{1+3i})(6-\sqrt{3}i) \qquad z_3 = \frac{5+i}{3-i} \qquad z_4 = (1-i)^5$$

4. Let $z_1 = 3+2i$, $z_2 = 1-i$. Plot in the complex plane $z_1, z_2, z_3 = z_1 + z_2, z_4 = z_1 - \bar{z}_2$, $z_5 = \bar{z}_1 z_2$.

Exercise 2

1. Give the polar form (or exponential form) of the following complex numbers:

$$z_1 = 1 + e^{i\theta} \qquad z_2 = 1 - e^{-i\theta} \qquad z_3 = e^{i\theta} + e^{i\phi} \qquad z_4 = \frac{1+e^{i\theta}}{1+e^{i\phi}}$$

2. Solve the equation $z^n = 1$ and plot the solutions for $n = 5$ in the complex plane.

3. Solve the equation $z^2 = \frac{1+i}{\sqrt{2}}$ and give the solutions in algebraic form. Deduce the value of $\cos\left(\frac{\pi}{8}\right)$ and $\sin\left(\frac{\pi}{8}\right)$.

Exercise 3

Find the scalar product $\langle \mathbf{v}, \mathbf{w} \rangle$ of the following pairs of vectors in $\mathbb{C}^d$:

1. $d = 2$, $\mathbf{v} = (1 + i, 2 + 3i)$, $\mathbf{w} = (4 - 2i, 3 + i)$;
2. $d = 2$, $\mathbf{v} = (4 - 2i, 3 + i)$, $\mathbf{w} = (1 + i, 2 + 3i)$;
3. $d = 3$, $\mathbf{v} = (2 - i, 1, -3i)$, $\mathbf{w} = (4, -i, 2 + i)$;
4. $d = 3$, $\mathbf{v} = (1 + 2i, 2 - i, 3 + i)$, $\mathbf{w} = (1, 2i, 3 - i)$.

Exercise 4

Check whether the following vectors in $\mathbb{C}^d$ form an orthonormal basis :

1. $d = 2$, $\mathbf{u}_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} \frac{1}{2} \\ -\frac{i}{2} \end{pmatrix}$;
2. $d = 2$, $\mathbf{u}_1 = \frac{1}{2} \begin{pmatrix} 1 + i \\ 1 - i \end{pmatrix}$, $\mathbf{u}_2 = \frac{1}{2} \begin{pmatrix} 1 - i \\ 1 + i \end{pmatrix}$;
3. $d = 3$, $\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{u}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ i \\ 1 \end{pmatrix}$, $\mathbf{u}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}$;
4. $d = 3$, $\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$, $\mathbf{u}_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 + i \\ 1 - i \\ 1 - i \end{pmatrix}$, $\mathbf{u}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -i \\ 2i \end{pmatrix}$.

Exercise 5

1. Prove that all eigenvalues $\lambda_1, \ldots, \lambda_d$ of arbitrary Hermitian matrix $H \in \mathbb{C}^{n \times n}$ are real.
2. Prove also that the eigenvectors of distinct eigenvalues are orthogonal.
3. Deduce that we can always choose the set of eignevectors as an orthonormal basis $v_1, \ldots, v_d$.
4. Finally deduce that for a Hermitian matrix we have the decomposition

$$H = \sum_{i=1}^{d} \lambda_i v_i v_i^\dagger$$

Exercise 6

Show that the matrix $A = \begin{pmatrix} 3 & 1 - i \\ 1 + i & 4 \end{pmatrix} \in \mathbb{C}^{2 \times 2}$ is Hermitian and find its eigenvalues and eigenvectors.

Exercise 7

Let U be an arbitrary unitary matrix in $\mathbb{C}^{n\times n}$.

1. Show that a unitary matrix preserves the scalar product: $(U\mathbf{w})^\dagger U\mathbf{z} = \mathbf{w}^\dagger \cdot \mathbf{z}$.
2. Show that a unitary matrix preserves the norm of a vector $\mathbf{z} \in \mathbb{C}^n$, $\|U\mathbf{z}\| = \|\mathbf{z}\|$.
3. Show that the lines of a unitary matrix form an orthonormal basis. Show that the columns also form an orthonormal basis.
4. Show that the modulus of any eigenvalue of U is 1.

Exercise 8

Show that matrix

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \theta \in \mathbb{R}$$

is unitary and find its eigenvalues and eigenvectors.