Solution Set 4

Solution 1: Laplace transform basics

(a) Determine the Laplace transform and the associated region of convergence for the following functions of time:

i. $x(t) = e^{-2t}u(t) - e^{3t}u(-t)\,,$ *ii.* $x(t) = \delta(t-t_0) - u(-t)\,,$ *iii.* $x(t) = te^{-3|t|}\,.$

(b) Determine the inverse Laplace transform of

$$X(s) = \frac{3s+5}{s^2+s-12}, \quad Re\{s\} > 3.$$

Solution

(a)

i. From Appendix 6.B we get

$$X(s) = \frac{1}{s+2} + \frac{1}{s-3} \tag{1}$$

and the ROC is $-2 < Re\{s\} < 3$, since the signal is two-sided.

ii. This part can be solved by looking at the Appendix 6.B, however we will solve it for completeness.

$$\begin{aligned}
X(s) &= \int_{-\infty}^{\infty} (\delta(t-t_0) - u(-t))e^{-st}dt &(2)\\
&= \int_{-\infty}^{\infty} \delta(t-t_0)e^{-st}dt - \int_{-\infty}^{\infty} u(-t)e^{-st}dt &(3)\\
&= e^{-st_0} - \int_{-\infty}^{0} e^{-st}dt &(4)\\
&= e^{-st_0} + \frac{e^{-st}}{s}\Big|_{-\infty}^{0} &(5)\\
&= e^{-st_0} + \frac{1}{s} &(6)
\end{aligned}$$

where (3) follows from linearity and (5) follows if $Re\{s\} < 0$.

iii. Let us try to rewrite the signal as $x(t) = te^{-3|t|} = te^{-3t}u(t) + te^{3t}u(-t)$. By using the Appendix 6.B we get

$$X(s) = \frac{1}{(s+3)^2} - \frac{1}{(s-3)^2} \tag{7}$$

where the ROC is $-3 < Re\{s\} < 3$. An alternative approach is to use the definition of Laplace transform.

(b) Let us start form partial fraction decomposition

$$\begin{aligned} X(s) &= \frac{3s+5}{s^2+s-12} && (8)\\ &= \frac{3s+5}{(s-3)(s+4)} && (9)\\ &= \frac{A}{s-3} + \frac{B}{s+4}, && (10) \end{aligned}$$

where we obtain $A = 2$ and $B = 1$. Taking into account the ROC, $Re\{s\} > 3$ the signal must be $x(t) = 2e^{3t}u(t) + e^{-4t}u(t)$.

Solution 2: The Region of Convergence

The Laplace transform for an absolutely integrable signal $x(t)$ is known to have a pole at $s = 2$. Answer the following questions and briefly justify your answer.

(a) Could $x(t)$ be of finite duration?

(b) Could $x(t)$ be left-sided?

(c) Could $x(t)$ be right-sided?

(d) Could $x(t)$ be two-sided?

Solution

The fact that $x(t)$ is absolutely integrable means that it has a Fourier transform and that the imaginary axis is in the ROC of the Laplace transform. Therefore, there are two possibilities, (i) the ROC extends from $Re(s) = 2$ to negative infinity or (ii) ROC extends from $Re(s) = 2$ to $Re(s) = b$ for some $b < 0$.

(a) No. A finite duration absolutely integrable signal would have the whole complex place as the ROC. It would not have any poles.

(b) Yes. If the ROC as in option (i), the signal is left-sided.

(c) No. A right-sided signal would have a right-sided ROC. With a pole at $s = 2$ this would mean that the imaginary axis in not in the ROC and this would contradict the fact that the signal is absolutely integrable.

(d) Yes. If the ROC as in option (ii), the signal is two-sided.

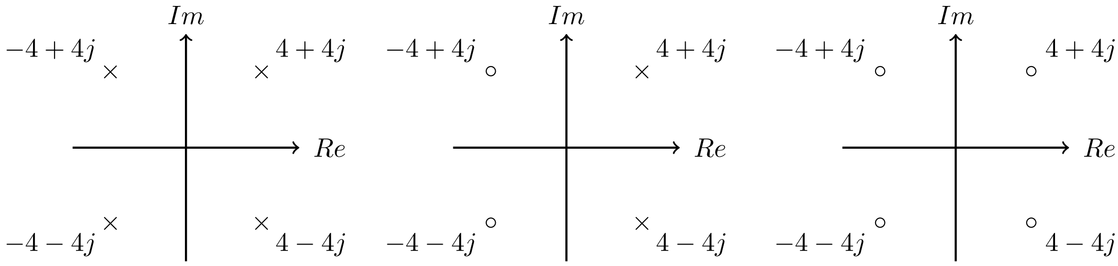
Solution 3: The Region of Convergence II

In this problem, we study the pole-zero plot of a signal $x(t)$.

(a) Figure 1 shows three different pole-zero plots. For *each* case separately, determine all possible ROCs.

(b) Suppose we know that $x(t)e^{5t}$ is absolutely integrable. Using this information about the signal, you can now identify which of the possible ROCs that you have found in Part *(a)* is the right one. Do this separately for each of the three cases.

(c) Suppose we know that $x(t) * (e^{2t}u(-t))$ is absolutely integrable, where $u(t)$ is the step function. Again, we can leverage this information about the signal to find the right ROC. Do this separately for each of the three cases.



(a) Case 1 (b) Case 2 (c) Case 3

Figure 1: Three possible pole-zero plots

Solution

(a) As the ROC cannot contain any pole, so for

- Case 1, there are three possible ROCs: $Re(s) < -4$ or $-4 < Re(s) < 4$ or $Re(s) > 4$;
- Case 2, there are two possible ROCs: $Re(s) < 4$ or $Re(s) > 4$;
- Case 3, the ROC is the entire s-plane.

(b) Let $X(s)$ denote the Laplace transform of $x(t)$ with ROC R. We know the Laplace transform of $x(t)e^{5t}$ is $X(s-5)$. The ROC R_1 of the new Laplace transform is R shifted by 5 to the right. If $x(t)e^{5t}$ is absolutely integrable, R_1 must contain the $j\omega$ axis. Hence for

- Case 1, this is possible if R is $Re(s) < -4$;
- Case 2, this is possible if R is $Re(s) < 4$;
- Case 3, R is the entire s-plane

(c) Let $X(s)$ denote the Laplace transform of $x(t)$ with ROC R.

We know that the Laplace transform of $e^{2t}u(-t)$ is $\frac{1}{2-s}$ with $Re(s) < 2$. The Laplace transform of $x(t) * (e^{2t}u(-t))$ is $\frac{X(s)}{2-s}$ with ROC $R_2 := R \cap \{Re(s) < 2\}$. If $x(t) * (e^{2t}u(-t))$ is absolutely integrable, then R_2 must contain the $j\omega$ axis. Hence for

- Case 1, this is possible if R is $-4 < Re(s) < 4$;
- Case 2, this is possible if R is $Re(s) < 4$;
- Case 3, R is the entire plane.

Solution 4: Transfer Function

Consider the system with impulse response

$$h(t) = \begin{cases} e^{-at}\cos(\omega_0 t) & t \geq 0, \\ -e^{bt}\sin(\omega_0 t), & t < 0, \end{cases}$$

where a, b and ω_0 are positive real numbers.

(a) Is this system stable?

(b) Find the transfer function $H(s)$ and the ROC.

(c) Draw the pole-plot for this transfer function. How many other systems do there exist that have exactly the same transfer function as the system considered here (but a different ROC)? Indicate the corresponding ROCs.

Solution:

(a) We should bound the following integral by a constant

$$\begin{aligned}\int_{-\infty}^{\infty} |h(t)|dt &= \int_{0}^{\infty} |e^{-at}\cos(\omega_0 t)|dt + \int_{-\infty}^{0} |-e^{bt}\sin(\omega_0 t)|dt \\ &\leq \int_{0}^{\infty} e^{-at}dt + \int_{-\infty}^{0} e^{bt}dt \\ &= \frac{1}{a} + \frac{1}{b}\end{aligned}$$

(b) First let us write impulse response as $h(t) = e^{-at}\cos(\omega_0 t)u(t) - e^{bt}\sin(\omega_0 t)u(-t)$ By using Appendix 7.B we obtain

$$\begin{aligned}H(s) &= \frac{s+a}{(s+a)^2+\omega_0^2} + \frac{\omega_0}{(s-b)^2+\omega_0^2} \\ &= \frac{s+a}{(s+a-j\omega_0)(s+a+j\omega_0)} + \frac{\omega_0}{(s-b+j\omega_0)(s-b-j\omega_0)}\end{aligned}$$

The Laplace transform is meaningful when $Re(s) > -a$ and $Re(s) < b$.

(c) The pole-plot for the above transfer function looks like the following

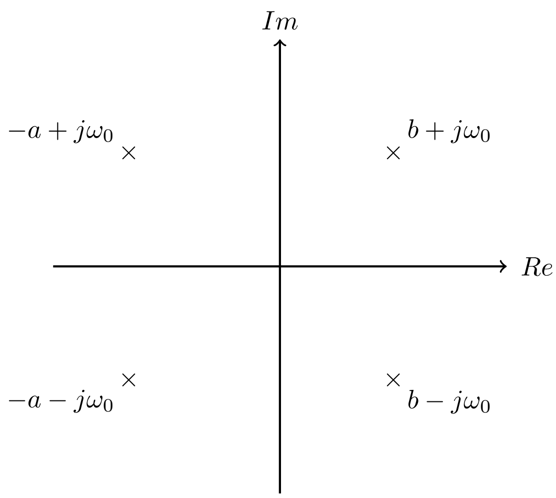


Figure 2: Pole-plot.

There are two other systems with the same transfer function but with a different ROC.

- The first system has ROC $Re(s) < -a$ with the corresponding impulse response

$$h(t) \quad = -(e^{-at}\cos(\omega_0 t) + e^{bt}\sin(\omega_0 t))u(-t) \tag{11}$$

where $u(t)$ denotes the step function. This system is unstable and non-causal (in fact anti-causal).

- The second system has ROC $Re(s) > b$ with the corresponding impulse response

$$h(t) \quad = (e^{-at}\cos(\omega_0 t) + e^{bt}\sin(\omega_0 t))u(t) \tag{12}$$

where $u(t)$ denotes the step function. This system is unstable but causal.

Solution 5: LTI Systems and Laplace transform

For a linear time-invariant system, it is known that the transfer function is given by

$$H(s) = \frac{2(s-2)}{(s+1)(s^2-2s+2)}.$$

As in lecture, we now study the pole-zero plot of this transfer function. That is, we study a transfer function that is a rational equation of two polynomials: $H(s) = \frac{P(s)}{Q(s)}$. We draw the roots of $Q(s)$ with a $\times$-symbol in the s-plane and we call them the *poles* of our system. Then, we draw the roots of $P(s)$ with an $\circ$-symbol and we call them the *zeros* of our system.

(a) Draw the pole/zero diagram for $H(s)$.

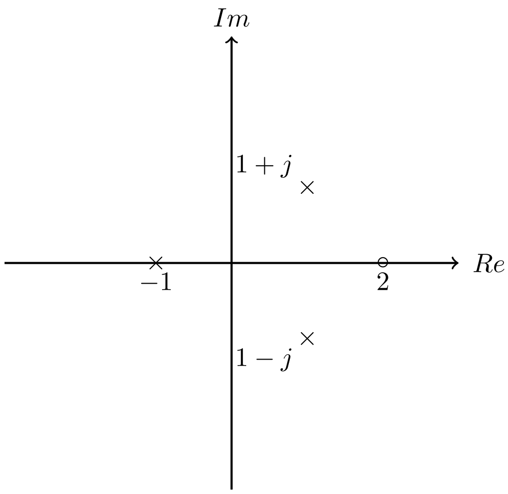
(b) Suppose that apart from $H(s)$, you are also told that the system is *causal.* Is the resulting system also stable?

(c) Determine the differential equation that describes this system.

Solution

(a)

$$\begin{aligned} H(s) &= \frac{2(s-2)}{(s+1)(s^2-2s+2)} \\ &= \frac{2(s-2)}{(s+1)(s-(1+j))(s-(1-j))} \end{aligned}$$



(b) Looking at our pole-zero plot, we conclude we can have either of the following Regions of Convergence:

- $Re(s) < -1$
- $-1 < Re(s) < 1$
- $1 < Re(s)$

Let us restate the following conditions for causality and stability. Assuming a rational transfer function

A system is

Causal	if and only if the region of convergence is the plane right of the rightmost pole.
Stable	if and only if the region of convergence contains the imaginary axis of the s-plane.
Causal and Stable	if and only if all poles lie in the left half of the s-plane and the region of convergence lies right of the rightmost pole.

At this point we can already conclude that the system can never be causal and stable at the same time. It can only have at most one of the two properties causal, but it's not stable. *(c)*

$$H(s) = \frac{2(s-2)}{(s+1)(s^2-2s+2)} = \frac{2(s-2)}{s^3-s^2+2} = \frac{Y(s)}{X(s)}$$

Cross-multiply and find:

$$\begin{aligned} 2(s-2)X(s) &= (s^3-s^2+2)Y(s) \\ &\Updownarrow \\ 2\frac{d}{dt}x(t) - 4x(t) &= \frac{d^3}{dt^3}y(t) - \frac{d^2}{dt^2}y(t) + 2y(t). \end{aligned}$$

Solution 6: Laplace transform and LTI systems

A causal LTI system $\mathcal{H}$ with inmpulse response $h(t)$ has its input $x(t)$ and output $y(t)$ related through a linear constant-coefficient differential equation of the form

$$\frac{d^3}{dt^3}y(t) + (1+\alpha)\frac{d^2}{dt^2}y(t) + \alpha(\alpha+1)\frac{d}{dt}y(t) + \alpha^2 y(t) = x(t).$$

(a) If

$$g(t) = \frac{d}{dt}h(t) + h(t),$$

how many poles does $G(s)$ have? What are they?

(b) For what real values of the parameter α is $\mathcal{H}$ guaranteed to be stable?

(c) For what real values of the parameter α is the $ROC(G(s))$ strictly bigger than the $ROC(H(s))$?

Solution:
(a) We can apply the differentiation in time property to the differential equation to find

$$H(s) = \frac{1}{s^3 + (1+\alpha)s^2 + \alpha(\alpha+1)s + \alpha^2}.$$

Applying the differentiation in time and linearity property again we find

$$G(s) = \frac{s+1}{s^3 + (1+\alpha)s^2 + \alpha(\alpha+1)s + \alpha^2} = \frac{1}{s^2 + \alpha s + \alpha^2} = \frac{1}{\left(s - \frac{1}{2}(-\alpha + j\alpha\sqrt{3})\right)\left(s - \frac{1}{2}(-\alpha - j\alpha\sqrt{3})\right)}.$$

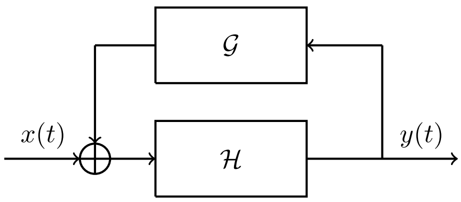
Thus, $G(s)$ has two poles: $s = \frac{1}{2}(-\alpha + j\alpha\sqrt{3})$ and $s = \frac{1}{2}(-\alpha - j\alpha\sqrt{3})$.

(b) For $\mathcal{H}$ to be stable, the $ROC(H(s))$ needs to contain the imaginary axis. This will happen if $\alpha > 0$

(c) Note that $H(s)$ has three poles: the same ones as $G(s)$ as well as one at $s = -1$. If $-\frac{\alpha}{2} < -1$, then the pole-zero cancelation above will enlarge the region of convergence for $G(s)$. Otherwise, the region of convergence will remain the same. So, the region gets enlarged for $\alpha > 2$.

Solution 7: Feedback system

In class we studied the feedback composition of two continuous-time systems and derived an equivalent transfer function in terms of $H(s)$ and $G(s)$.



Suppose

$$H(s) = \frac{1}{s-1} \quad \text{and} \quad G(s) = b - s,$$

with the the overall feedback system assumed to be causal. For what real values of b is the feedback system also stable?

Solution: From we the feedback system, in the Laplace domain we can write the following relation

$$Y(s) = H(s)[X(s) + G(s)Y(s)], \tag{13}$$

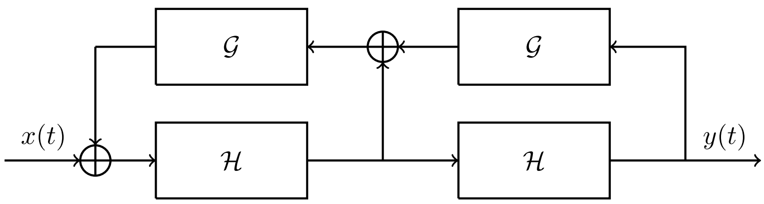
thus, $Y(s)[1 - H(s)G(s)] = H(s)X(s)$. Therefore, the overall transfer function is

$$\begin{aligned} H_{\text{overall}} &= \frac{Y(s)}{X(s)} && (14)\\ &= \frac{H(s)}{1 - H(s)G(s)} && (15)\\ &= \frac{\frac{1}{s-1}}{1 - \frac{b-s}{s-1}} && (16)\\ &= \frac{1}{2s - 1 - b}. && (17) \end{aligned}$$

The overall system has a single pole at $\frac{1+b}{2}$ and since the system is causal the ROC is $Re\{s\} > \frac{1+b}{2}$. In order for the system to be stable the ROC should include the imaginary axis, e.g. $\frac{1+b}{2} < 0$, or $b < -1$.

Solution 8: Transfer function of a composite system

(a) Consider the continuous-time system interconnect shown as follows, where the component systems $\mathcal{G}$ and $\mathcal{H}$ are LTI systems with transfer functions $G(s)$ and $H(s)$, respectively. Derive the overall transfer function of the system as a function of $G(s)$ and $H(s)$.



Solution: Let's define two auxiliary signals $w(t)$ and $v(t)$ shown as Figure 3:

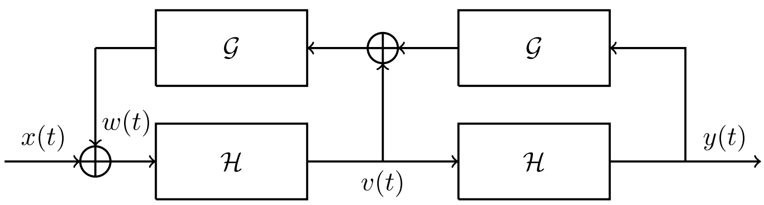


Figure 3: Interconnected LTI Systems:

It is easy to see that:

$$\begin{aligned} V(s) &= H(s)(X(s) + W(s)) \\ W(s) &= G(s)(V(s) + G(s)Y(s)) \\ Y(s) &= H(s)V(s) \end{aligned}$$

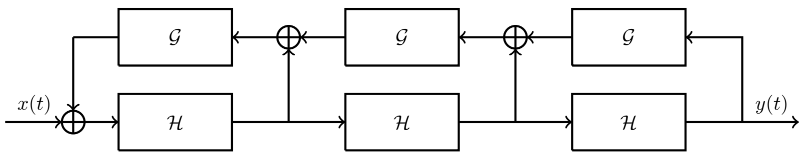
By eliminating $W(s)$ and $V(s)$, we can get

$$Y(s) = H(s)^2 \left(X(s) + \frac{G(s)H(s)X(s) + G(s)^2 Y(s)}{1 - G(s)H(s)} \right)$$

If we divide this expression by $X(s)$ and we solve for $Y(s)/X(s)$, we get

$$\frac{Y(s)}{X(s)} = \frac{H(s)^2}{1 - H(s)G(s) - H(s)^2 G(s)^2}.$$

(b)(Optional) The interconnected system from Part *(a)* could be thought of as having $K = 2$ stages. We can extend this to an arbitrary length — the following figure shows the case of $K = 3$ stages. For arbitrary K, find the formula for the overall transfer function of the system as a function of $G(s)$ and $H(s)$.



Let's define some auxiliary signals as Figure 4.

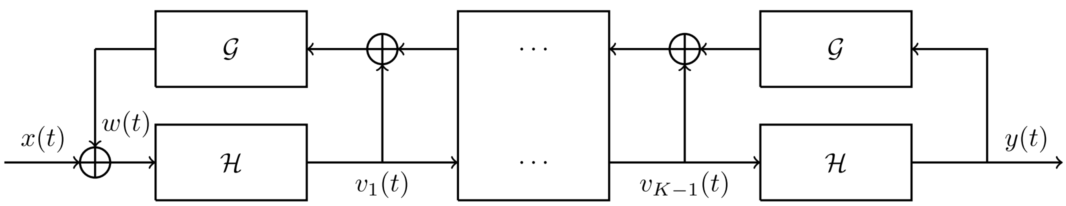


Figure 4: Interconnected LTI Systems with auxiliary signals

It is easy to see that

$$\begin{aligned} Y(s) &= H(s)^K (X(s) + W(s)) \\ Y(s) &= V_i(z) H(s)^{K-i} \\ W(s) &= \sum_{i=1}^{K} V_i(z) G(s)^i. \end{aligned}$$

By eliminating $W(s)$ and $V_i(s)$-s we can get

$$\begin{aligned} Y(s) &= H(s)^K (X(s) + \sum_{i=1}^{K} V_i(s) G(s)^i) \\ &= H(s)^K (X(s) + \sum_{i=1}^{K} \frac{Y(s)}{H(s)^{K-i}} G(s)^i) = H(s)^K X(s) + \sum_{i=1}^{K} G(s)^i H(s)^i Y(s). \end{aligned}$$

Therefore the transfer function of the system with K stages is

$$\frac{Y(s)}{X(s)} = \frac{H(s)^K}{1 - \left(\sum_{i=1}^{K} G(s)^i H(s)^i \right)}.$$