

Problem Set 4

Problem 1: Laplace transform basics

(a) Determine the Laplace transform and the associated region of convergence for the following functions of time:

i. $x(t) = e^{-2t}u(t) - e^{3t}u(-t)$, ii. $x(t) = \delta(t - t_0) - u(-t)$, iii. $x(t) = te^{-3|t|}$.

(b) Determine the inverse Laplace transform of

$$X(s) = \frac{3s + 5}{s^2 + s - 12}, \quad \text{Re}\{s\} > 3.$$

Problem 2: The Region of Convergence

The Laplace transform for an absolutely integrable signal $x(t)$ is known to have a pole at $s = 2$. Answer the following questions and briefly justify your answer.

- (a) Could $x(t)$ be of finite duration?
- (b) Could $x(t)$ be left-sided?
- (c) Could $x(t)$ be right-sided?
- (d) Could $x(t)$ be two-sided?

Problem 3: The Region of Convergence II

In this problem, we study the pole-zero plot of a signal $x(t)$.

(a) Figure 1 shows three different pole-zero plots. For *each* case separately, determine all possible ROCs.

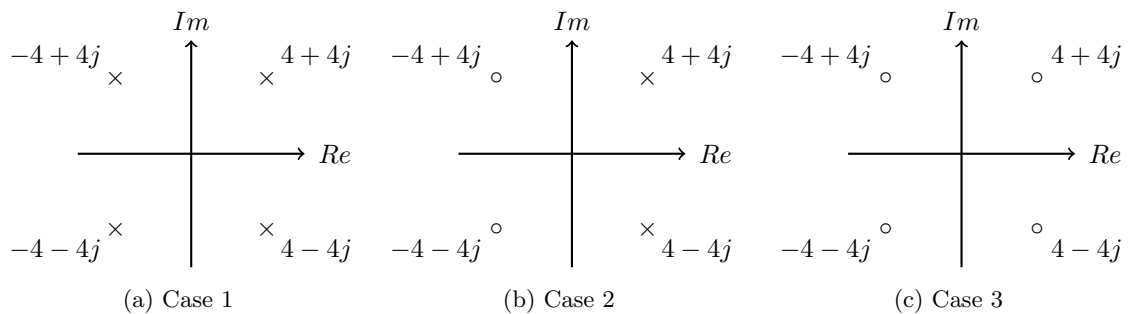


Figure 1: Three possible pole-zero plots

(b) Suppose we know that $x(t)e^{5t}$ is absolutely integrable. Using this information about the signal, you can now identify which of the possible ROCs that you have found in Part (a) is the right one. Do this separately for each of the three cases.

(c) Suppose we know that $x(t) * (e^{2t}u(-t))$ is absolutely integrable, where $u(t)$ is the step function. Again, we can leverage this information about the signal to find the right ROC. Do this separately for each of the three cases.

Problem 4: Transfer Function

Consider the system with impulse response

$$h(t) = \begin{cases} e^{-at} \cos(\omega_0 t) & t \geq 0, \\ -e^{bt} \sin(\omega_0 t), & t < 0, \end{cases}$$

where a, b and ω_0 are positive real numbers.

(a) Is this system stable?

(b) Find the transfer function $H(s)$ and the ROC.

(c) Draw the pole-plot for this transfer function. How many other systems do there exist that have exactly the same transfer function as the system considered here (but a different ROC)? Indicate the corresponding ROCs.

Problem 5: LTI Systems and Laplace transform

For a linear time-invariant system, it is known that the transfer function is given by

$$H(s) = \frac{2(s-2)}{(s+1)(s^2-2s+2)}.$$

As in lecture, we now study the pole-zero plot of this transfer function. That is, we study a transfer function that is a rational equation of two polynomials: $H(s) = \frac{P(s)}{Q(s)}$. We draw the roots of $Q(s)$ with a $\times$ -symbol in the s -plane and we call them the *poles* of our system. Then, we draw the roots of $P(s)$ with an $\circ$ -symbol and we call them the *zeros* of our system.

(a) Draw the pole/zero diagram for $H(s)$.

(b) Suppose that apart from $H(s)$, you are also told that the system is *causal*. Is the resulting system also stable?

(c) Determine the differential equation that describes this system.

Problem 6: Laplace transform and LTI systems

A causal LTI system $\mathcal{H}$ with impulse response $h(t)$ has its input $x(t)$ and output $y(t)$ related through a linear constant-coefficient differential equation of the form

$$\frac{d^3}{dt^3}y(t) + (1 + \alpha)\frac{d^2}{dt^2}y(t) + \alpha(\alpha + 1)\frac{d}{dt}y(t) + \alpha^2y(t) = x(t).$$

(a) If

$$g(t) = \frac{d}{dt}h(t) + h(t),$$

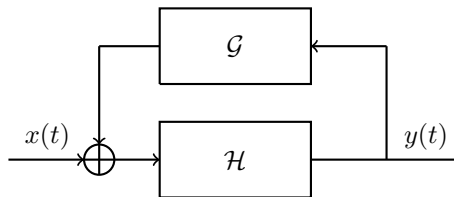
how many poles does $G(s)$ have? What are they?

(b) For what real values of the parameter α is $\mathcal{H}$ guaranteed to be stable?

(c) For what real values of the parameter α is the $ROC(G(s))$ strictly bigger than the $ROC(H(s))$?

Problem 7: Feedback system

In class we studied the feedback composition of two continuous-time systems and derived an equivalent transfer function in terms of $H(s)$ and $G(s)$.



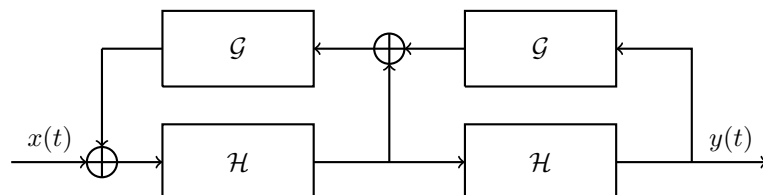
Suppose

$$H(s) = \frac{1}{s-1} \quad \text{and} \quad G(s) = b-s,$$

with the the overall feedback system assumed to be causal. For what real values of b is the feedback system also stable?

Problem 8: Transfer function of a composite system

(a) Consider the continuous-time system interconnect shown as follows, where the component systems $\mathcal{G}$ and $\mathcal{H}$ are LTI systems with transfer functions $G(s)$ and $H(s)$, respectively. Derive the overall transfer function of the system as a function of $G(s)$ and $H(s)$.



(b)(Optional) The interconnected system from Part (a) could be thought of as having $K = 2$ stages. We can extend this to an arbitrary length — the following figure shows the case of $K = 3$ stages. For arbitrary K , find the formula for the overall transfer function of the system as a function of $G(s)$ and $H(s)$.

