Fundamentals of electrical circuits and systems I
Prof. Jean-Philippe Thiran October 07, 2025

Solution Set 3

Solution 1: Hard and Easy

(a) A discrete-time sequence $x[n]$ is given as

$$x[n] = x_1[n] - x_1[n+3]$$

where

$$x_1[n] = \left(\frac{1}{2}\right)^n u[n]$$

Find $X(w)$.

(b) Compute the value of $\int_{-\pi}^{+\pi} X(w)dw$.

Hint: you could do it the hard way, or the easy way. We recommend finding the easy way!

Solution: (a) By applying the linearity and shift in time properties, we obtain

$$X(\omega) = X_1(\omega) - e^{3j\omega}X_1(\omega).$$

Since

$$X_1(\omega) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

we have that

$$X(\omega) = \frac{1 - e^{3j\omega}}{1 - \frac{1}{2}e^{-j\omega}}$$

(b) We have that,

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega.$$

We substitute $n = 0$,

$$x[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)d\omega$$

Thus,

$$\int_{-\pi}^{\pi} X(\omega)d\omega = 2\pi x[0] = \frac{14\pi}{8}$$

Solution 2: Frequency response

(a) Consider a continuous-time LTI system with frequency response $H(\omega)$ and real and even impulse response $h(t)$ (that is, $h(t) = h(-t)$). Suppose that we apply an input $x(t) = \sin(\omega_0 t)$ to this system. The resulting output can be shown to be of the form

$$y(t) = A\sin(\omega_0 t)$$

where A is a real number. Find A as a function of $H(\omega)$, and ω_0. For full credit do this without taking a Fourier Transform of $x(t)$.

(b) Consider a continuous-time LTI system with frequency response $H(\omega) = |H(\omega)|e^{j\arg H(\omega)}$ and real impulse response $h(t)$. Suppose that we apply an input $x(t) = \cos(\omega_0 t + \phi_0)$ to this system. The resulting output can be shown to be of the form

$$y(t) = Ax(t - t_0)$$

where A is a nonnegative real number representing an amplitude-scaling and t_0 is a time delay. Find A and t_0 as a function of $H(\omega)$, ω_0, and ϕ_0. For full credit do this without taking a Fourier Transform of $x(t)$.

Solution:

(a) Using Euler's formula we can write

$$x(t) = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right).$$

Since the impulse response is real and even, so is the frequency response: that is $H(\omega) = H(-\omega)$.

Then,

$$\begin{aligned} y(t) &= \frac{1}{2j}\left(H(\omega_0)e^{j\omega_0 t} - H(-\omega_0)e^{-j\omega_0 t}\right) \\ &= \frac{1}{2j}\left(H(\omega_0)e^{j\omega_0 t} - H(\omega_0)e^{-j\omega_0 t}\right) \\ &= H(\omega_0)\frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) \\ &= H(\omega_0)\sin(\omega_0 t) \end{aligned}$$

and $A = H(\omega_0)$.

(b) Using Euler's formula we can write

$$\begin{aligned} x(t) &= \frac{1}{2}\left(e^{j\omega_0 t + j\phi_0} + e^{-j\omega_0 t - j\phi_0}\right) \\ &= \frac{1}{2}\left(e^{j\phi_0}e^{j\omega_0 t} + e^{-j\phi_0}e^{-j\omega_0 t}\right) \end{aligned}$$

Since the impulse response is real, the frequency response satisfies $H(\omega) = H^*(-\omega)$. Then,

$$\begin{aligned} y(t) &= \frac{1}{2}\left(H(\omega_0)e^{j\phi_0}e^{j\omega_0 t} + H(-\omega_0)e^{-j\phi_0}e^{-j\omega_0 t}\right) \\ &= \frac{1}{2}\left(H(\omega_0)e^{j\phi_0}e^{j\omega_0 t} + H^*(\omega_0)e^{-j\phi_0}e^{-j\omega_0 t}\right) \\ &= |H(\omega_0)|\frac{1}{2}\left(e^{j\phi_0 + j\arg H(\omega_0)}e^{j\omega_0 t} + e^{-j\phi_0 - j\arg H(\omega_0)}e^{-j\omega_0 t}\right) \\ &= |H(\omega_0)|\cos(\omega_0 t + \arg H(\omega_0) + \phi_0) \end{aligned}$$

and so $A = |H(\omega_0)|$ and $t_0 = -\frac{\arg H(\omega_0)}{\omega_0}$.

Solution 3: Ideal filters

(a) A continuous-time ideal low-pass filter has a frequency response

$$H(\omega) = \begin{cases} 1, |\omega| \le \frac{\pi}{5} \\ 0, \text{ otherwise} \end{cases}$$

Find the output when the filter is applied to each of the following inputs

$$x(t) = \cos\frac{\pi}{3}t, \quad x(t) = 1, \quad x(t) = \text{sinc}\left(\frac{t}{6}\right), \quad x(t) = \text{sinc}\left(\frac{t}{4}\right). \tag{1}$$

(b) A discrete-time ideal low-pass filter has a frequency response

$$H(\omega) = \begin{cases} 1, |\omega| \le \frac{\pi}{5} \\ 0, \text{ otherwise} \end{cases}$$

on the interval $-\pi < \omega \le \pi$. Find the output when the filter is applied to each of the following inputs

$$x[n] = \cos\frac{13\pi}{6}n, \quad x[n] = (-1)^n, \quad x[n] = \delta[n]. \tag{2}$$

Solution:

(a)

Taking the Fourier Transform of $x(t) = \cos\frac{\pi}{3}t$ we note that $X(\omega)$ is two delta pulses at $\omega = \frac{\pi}{3}$ and $\omega = -\frac{\pi}{3}$. Then $H(\omega)X(\omega) = 0$ and so $y(t) = 0$.

Taking the Fourier Transform of $x(t) = 1$ we note that $X(\omega)$ is a delta pulse at $\omega = 0$. Then $H(\omega)X(\omega) = X(\omega)$ and so $y(t) = 1$ for all t.

From Appendix 4.B we see that the Fourier Transform of $x(t) = \text{sinc}\left(\frac{t}{6}\right)$ is non-zero only for $|\omega| \le \frac{\pi}{6}$. Then $H(\omega)X(\omega) = X(\omega)$ and so $y(t) = \text{sinc}\left(\frac{t}{6}\right)$.

From Appendix 4.B we see that the Fourier Transform of $x(t) = \text{sinc}\left(\frac{t}{4}\right)$ is itself a low-pass filter with gain 4.That is

$$X(\omega) = \begin{cases} 4, |\omega| \le \frac{\pi}{4} \\ 0, \text{ otherwise} \end{cases}$$

Then

$$Y(\omega) = H(\omega)X(\omega) = \begin{cases} 4, |\omega| \le \frac{\pi}{5} \\ 0, \text{ otherwise} \end{cases}$$

Taking the inverse Fourier Transform gives $y(t) = \frac{4}{5}\text{sinc}\left(\frac{t}{5}\right)$.

(b)

Note that $x[n] = \cos\frac{13\pi}{6}n = \cos\frac{\pi}{6}n$. Taking the Fourier Transform of $x[n]$ again produces two delta pulses at $\omega = \frac{\pi}{6}$ and $\omega = -\frac{\pi}{6}$ on the interval $-\pi < \omega \leq \pi$. Then $H(\omega)X(\omega) = X(\omega)$ and so $y(t) = \cos\frac{\pi}{6}n$.

Note again that $x[n] = (-1)^n = e^{j\pi n}$. Taking the Fourier Transform of $x[n]$ again produces a pulse at $\omega = \pi$ on the interval $-\pi < \omega \leq \pi$. Then $H(\omega)X(\omega) = 0$ and so $y(t) = 0$.

The Fourier Transform of $x[n] = \delta[n]$ is $X(\omega) = 1$. Then $Y(\omega) = H(\omega)X(\omega) = H(\omega)$. Taking the inverse transform yields $y[n] = \frac{1}{5}\text{sinc}(\frac{n}{5})$

Solution 4: Composition of systems

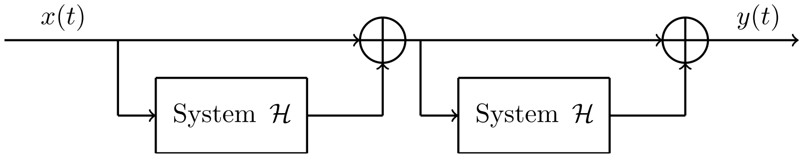

Figure 1: A composed system.

In this problem, we study the system composition illustrated in Figure 1 with input $x(t)$ and output $y(t)$, where we **assume** that the system $\mathcal{H}$ is known to be LTI and stable.

(a) Prove that the composed system shown in the figure is stable.

(b) Give the frequency response of the overall system in Figure 1 with input $x(t)$ and output $y(t)$ in terms of the frequency response $H(\omega)$ of the component system $\mathcal{H}$.

(c) For the special case where the system $\mathcal{H}$ is the LTI system with impulse response $h(t) = e^{-|t|}$, give an explicit formula for the frequency response of the overall system (expressed as a ratio of two polynomials in ω).

Solution

(a)

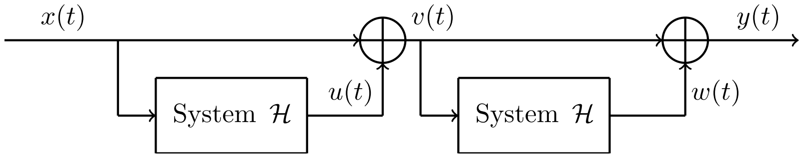

Figure 2: Labeled figure

Assume that $|x(t)| < B$ for all t and $B < \infty$. Then $|u(t)| < C$ because the system is stable thus,

$$\begin{aligned} |v(t)| &= |u(t) + x(t)| \\ &\leq |u(t)| + |x(t)| < B + C \end{aligned}$$

the signal $v(t)$ is bounded. Also $|w(t)| < D$ because the system $\mathcal{H}$ is stable so

$$\begin{aligned} |y(t)| &= |w(t) + v(t)| \\ &\leq |w(t)| + |v(t)| < B + C + D. \end{aligned}$$

(b) From the figure above we can write the output in terms of input as follows

$$y(t) = v(t) + (v * h)(t)$$

and the frequency domain we have $Y(\omega) = (1 + H(\omega))V(\omega)$. Similarly, $V(\omega) = (1 + H(\omega))X(\omega)$, thus the overall frequency response is $(1 + H(\omega))^2$.

(c) The frequency response of system $\mathcal{H}$ is

$$H(\omega) = \int_{-\infty}^{\infty} e^{-|t|}e^{-j\omega t}dt = \int_{-\infty}^{0} e^{(1-j\omega)t}dt + \int_{0}^{\infty} e^{-(1+j\omega)t}dt = \frac{1}{1+j\omega} + \frac{1}{1-j\omega} = \frac{2}{1+\omega^2}.$$

The overall frequency response will be

$$(1 + \frac{2}{1+\omega^2})^2 = \frac{9 + \omega^4 + 6\omega^2}{1 + \omega^4 + 2\omega^2}.$$

Solution 5: Step response (Review)

During the course we have seen as the impulse response completely characterizes an LTI system. Unfortunately, it is rather difficult to produce an infinite amplitude pulse with infinitesimal duration in practice. In this problem we define another type of response called the *step response* which is often used to characterize an LTI system. It is defined, in continuous time, as:

$$s(t) = u(t) * h(t)$$

That is, $s(t)$ characterizes how an LTI system reacts to the signal $u(t)$.

(a) An LTI system is known to have an impulse response

$$h(t) = e^{-\alpha t} u(t - t_0)$$

where $\alpha, t_0 > 0$. Find its step response.

(b) Use the convolution integral to show

$$h(t) = \frac{ds(t)}{dt} = s'(t).$$

(c) Recall that input-output relationship of a stable LTI system could be related by the equation

$$Y(\omega) = H(\omega)X(\omega)$$

where $Y(\omega)$ is the Fourier transform for the output signal $y(t)$, $X\omega)$ is the Fourier transform for the input signal $x(t)$, and $H(\omega)$ is the Fourier transform for the impulse response $h(t)$.

Assume that the Fourier Transform of $s(t)$ of an LTI system exists. Find the same input-output relationship in the frequency domain in terms of $S(\omega)$. That is, you need to derive the formula that relates $X(\omega)$ to $Y(\omega)$ in terms of $S(\omega)$.

Solution:

(a)

$$\begin{aligned}
s(t) &= u(t) * h(t) \\
&= \int_{-\infty}^{\infty} h(\tau) u(t - \tau) d\tau \\
&= \int_{-\infty}^{t} h(\tau) d\tau \\
&= \int_{-\infty}^{t} e^{-\alpha\tau} u(\tau - t_0) d\tau \\
&= \int_{t_0}^{t} e^{-\alpha\tau} d\tau
\end{aligned}$$

If $t < t_0$ then $s(t) = 0$. Otherwise

$$s(t) = -\frac{1}{\alpha}\left(e^{-\alpha t} - e^{-\alpha t_0}\right)$$

and the final answer is

$$s(t) = \frac{1}{\alpha}\left(e^{-\alpha t_0} - e^{-\alpha t}\right) u(t - t_0).$$

(b) Writing out the convolution integral

$$\begin{aligned} s(t) &= u(t) * h(t) \\ &= \int_{-\infty}^{\infty} h(\tau) u(t-\tau) d\tau \end{aligned}$$

then

$$\begin{aligned} \frac{ds(t)}{dt} &= \frac{d}{dt}\int_{-\infty}^{t} h(\tau) d\tau \\ &= h(t) \end{aligned}$$

where the last line follow from the fundamental theorem of calculus.

(c) From part (b) we have that $h(t) = \frac{ds(t)}{dt}$ and using differentiation in time property of Fourier Transform gives $H(\omega) = j\omega S(\omega)$. Combining this with the equation $Y(\omega) = H(\omega)X(\omega)$ yields $Y(\omega) = j\omega S(\omega)X(\omega)$.

Solution 6: A simple communication system

Many communication systems, for example mobile phones and other wireless devices, send information across free space using electromagnetic waves. To send these electromagnetic waves across long distances, the frequency of the transmitted signal must be very high compared to the frequency of the information signal. An essential technique in designing such communication systems is called *modulation*. During modulation an information signal to be transmitted is embedded, or modulated, onto a higher frequency waveform called carrier. In this problem we analyze a simple communication system that uses the principle of modulation.

Let $x(t)$ be a real-valued signal for which $X(\omega) = 0$ when $|\omega| \geq 2000\pi$. In order to communicate $x(t)$ over free space modulation is performed to produce the transmitted signal $g(t)$, where

$$g(t) = x(t) \cos 2000\pi t.$$

(a) Find the Fourier transform, $G(\omega)$, of the transmitted signal $g(t)$.

Once the signal $g(t)$ is received, it needs to be processed (demodulated) to recover $x(t)$. A proposed demodulation system is illustrated in Figure 5 where $\mathcal{H}$ is an *ideal low pass filter* with a frequency response given by

$$H(\omega) = \begin{cases} b, & |\omega| \leq \omega_c \\ 0, & \text{elswehere.} \end{cases}$$

(b) Find the Fourier transform, $Z(\omega)$, of the input to the low pass filter $z(t)$.

(c) Find the gain b and the cut-off frequency ω_c such that $y(t) = x(t)$.

Solution: *(a)*

Applying the convolution in frequency property from Appendix 4.A and the Fourier transform pair for cosine form Appendix 4.B we obtain

$$\begin{aligned} G(\omega) &= \frac{1}{2\pi} X(\omega) * \pi(\delta(\omega - 2000\pi) + \delta(\omega + 2000\pi)) \\ &= \frac{1}{2}(X(\omega - 2000\pi) + X(\omega + 2000\pi)). \end{aligned}$$

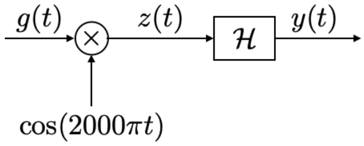

Figure 3: Proposed demodulation system.

(b)

Again, applying the convolution in frequency property from Appendix 4.A and the Fourier transform pair for cosine form Appendix 4.B we obtain

$$\begin{aligned} Z(\omega) &= \frac{1}{2\pi} G(\omega) * \pi(\delta(\omega - 2000\pi) + \delta(\omega + 2000\pi)) \\ &= \frac{1}{2}(G(\omega - 2000\pi) + G(\omega + 2000\pi)) \\ &= \frac{1}{4}(X(\omega - 4000\pi) + 2X(\omega) + X(\omega + 4000\pi)) \end{aligned}$$

(c)

By looking at the expression for $Z(\omega)$ we see that $b = 2$ and $\omega_c = 2000\pi$ will produce the desired result. It might be helpful to sketch the Fourier transform of $Z(\omega)$ in terms of the Fourier transform of $X(\omega)$ to see what is going on in the frequency domain.

Solution 7: Sampling sinusoids

(a) A sampling system samples continuous-time signals with frequency $\omega_s = 1000\pi$. According to the sampling theorem, which of the following signals could be reconstructed exactly if sampled by this system:

i. $x(t) = \cos 200\pi t$, *ii.* $x(t) = \sin 450\pi t$, *iii.* $x(t) = \sin 2500\pi t$, *iv.* $x(t) = \cos 200\pi t + \sin 800\pi t$.

(b) A sampling system samples continuous-time signals with sampling interval $T = 0.5 \times 10^{-3}$. According to the sampling theorem, which of the following signals could be reconstructed exactly if sampled by this system:

i. $x(t) = \cos 1000\pi t$, *ii.* $x(t) = \sin 2500\pi t$, *iii.* $x(t) = \cos 500\pi t + \sin 300\pi t$, *iv.* $x(t) = \sin 1500\pi t$.

Solution:

(a) We know that one way which is sufficient to reconstruct the original signal is to choose $\omega_s > 2\omega_M$ (the frequency such that $X(\omega) = 0$ for $|\omega| > \omega_M$) and apply a low pass filter with cut-off frequency ω_M. Respectively for:

i. $x(t) = \cos 200\pi t$, $X(\omega) = \pi\delta(\omega - 200\pi) + \pi\delta(\omega + 200\pi)$. So $\omega_M = 200\pi$ and $\omega_s > 2\omega_M$, thus YES we can reconstruct it exactly.

ii. $x(t) = \sin 450\pi t$, $X(\omega) = \frac{\pi}{j}\delta(\omega - 450\pi) - \frac{\pi}{j}\delta(\omega + 450\pi)$. So $\omega_M = 450\pi$ and $\omega_s > 2\omega_M$, thus YES we can reconstruct it exactly.

iii. $x(t) = \sin 2500\pi t$, $\omega_M = 2500\pi$ and $\omega_s < 2\omega_M$, thus NO we cannot reconstruct it.

iv. $x(t) = \cos 200\pi t + \sin 800\pi t$, $X(\omega) = \pi\delta(\omega - 200\pi) + \pi\delta(\omega + 200\pi) + \frac{\pi}{j}\delta(\omega - 800\pi) - \frac{\pi}{j}\delta(\omega + 800\pi)$

So $\omega_M = \max\{200\pi, 800\pi\} = 800\pi$ and $\omega_s < 2\omega_M$, thus NO we cannot reconstruct it.

(b) The sampling frequency is $\omega_s = \frac{2\pi}{T} = 4000\pi$. Then, it becomes similar to part (a).

i. $x(t) = \cos 1000\pi t$, $\omega_M = 1000\pi$ and $\omega_s > 2\omega_M$, thus YES we can reconstruct it exactly.

ii. $x(t) = \cos 2500\pi t$, $\omega_M = 2500\pi$ and $\omega_s < 2\omega_M$, thus NO we cannot reconstruct it.

iii. $x(t) = \cos 500\pi t + \sin 300\pi t$, $\omega_M = 500\pi$ and $\omega_s > 2\omega_M$, thus YES we can reconstruct it exactly.

iv. $x(t) = \sin 1500\pi t$, $\omega_M = 1500\pi$ and $\omega_s > 2\omega_M$, thus YES we can reconstruct it exactly.

Solution 8: Sampling sinusoids - Part 2

A mystery signal $x(t)$ is sampled with frequency $\omega_s = 1000\pi$ using impulse-train sampling, and then reconstructed with a low-pass filter with cut-off frequency $\omega_c = 500\pi$. The reconstructed signal is

$$x_r(t) = \cos 200\pi t.$$

We do not know anything else about $x(t)$. Which of the following signals could be $x(t)$?

i. $x(t) = \cos 300\pi t$, *ii.* $x(t) = \cos 200\pi t$, *iii.* $x(t) = \cos 1200\pi t$, *iv.* $x(t) = \cos 800\pi t$.

Make sure to justify your answers for full credit.

Solution:

(a) From equation (5.23) in the lecture notes we have

$$X_p(\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s). \tag{3}$$

and from equation (5.34) in the lecture notes we have

$$X_r(\omega) = X_p(\omega)H(\omega). \tag{4}$$

i. If $x(t) = \cos(300\pi t)$, then $\omega_s > 2\omega_M$, then there is no aliasing and after the low pass filter we get $x_r(t) = \cos(300\pi t) \neq \cos(200\pi t)$.

ii. If $x(t) = \cos(200\pi t)$, then $\omega_s > 2\omega_M$, then there is no aliasing and after the low pass filter we get $x_r(t) = \cos(200\pi t)$.

iii. If $x(t) = \cos(1200\pi t)$, then $\omega_s < 2\omega_M$, then there is aliasing so the sampling theorem do not apply. After we apply (3) and (4) we get $x_r(t) = \cos(200\pi t)$, check the figure below.

iv. If $x(t) = \cos(800\pi t)$, then $\omega_s < 2\omega_M$, then there is aliasing so the sampling theorem do not apply. After we apply (3) and (4) we get $x_r(t) = \cos(200\pi t)$, it has a similar figure as part iii.

Solution 9: Nyquist rate

Let $x(t)$ be a signal with Nyquist rate ω_0. Determine the Nyquist rate for each of the following signals:

(a) $x(t) - x(t-1)$

(b) $x(t)x(t-1)$

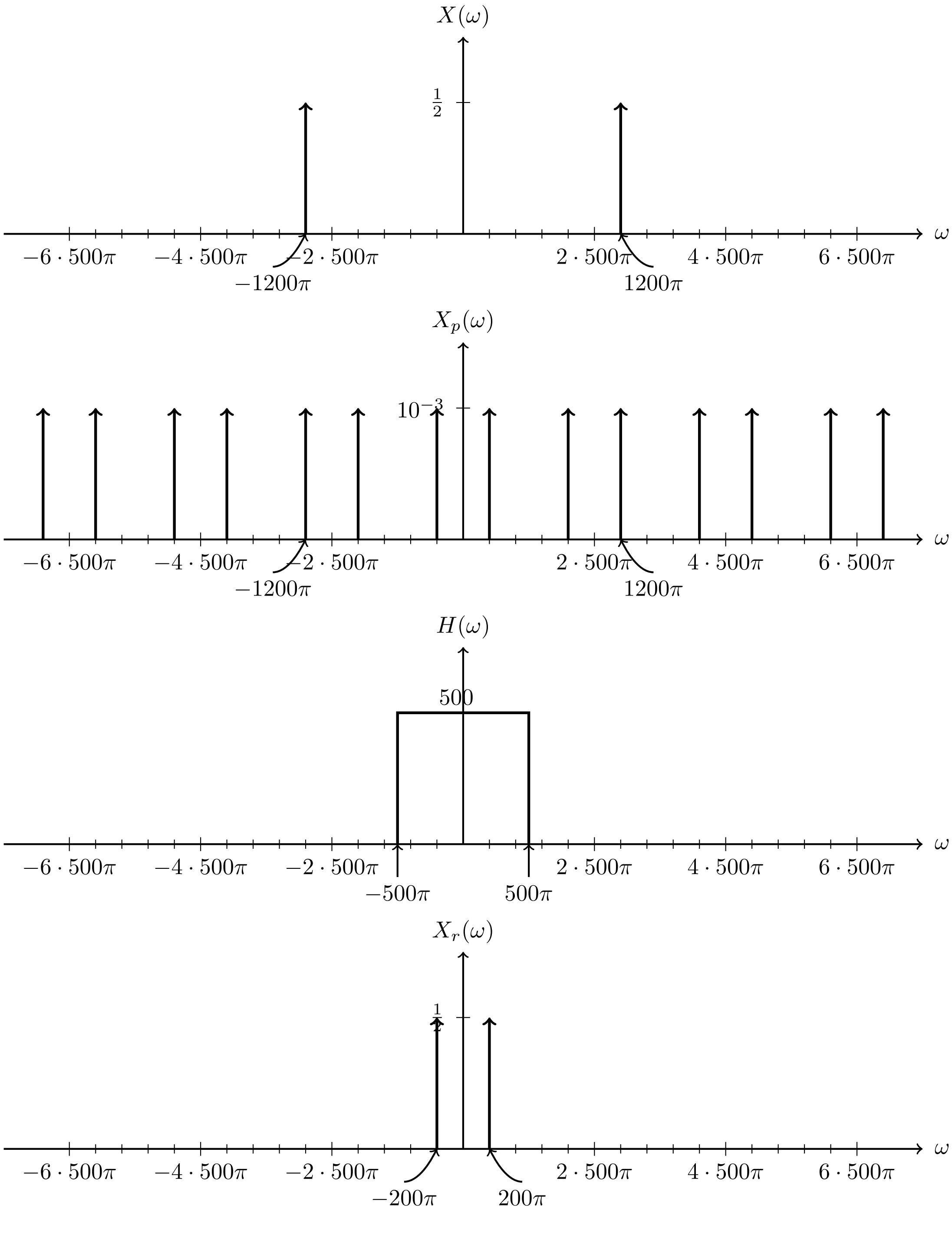

Figure 4: For part *iii.* of (*c*).

(c) $(x * z)(t)$ where $z(t) = \sin \frac{\omega_0}{3} t$
Hint: A complete answer should include two possibilities.

(d) $(x * z)(t)$ where $z(t) = \cos \omega_0 t$

Solution:

(a) For the sake of simplicity the alternative interpretation of "$x(t)$ be a signal with Nyquist rate ω_0" is: $X(\omega) = 0$ for $|\omega| > \frac{\omega_0}{2}$. The signal $y(t) = x(t) - x(t-1)$, in the frequency domain will be $Y(\omega) = X(\omega) - e^{-j\omega}X(\omega)$. In other words $X(\omega)$ is superposed with its own shifted copy, however $Y(\omega) = 0$ for $|\omega| > \frac{\omega_0}{2}$ and the Nyquist rate remains the same, ω_0.

(b) The signal $y(t) = x(t)x(t-1)$ in the frequency domain will be $Y(\omega) = \frac{1}{2\pi}(X(\omega) * (e^{-j\omega}X(\omega)))$. So if $X(\omega) = 0$ for $|\omega| > \frac{\omega_0}{2}$, then the scaled version of the convolution with itself which is $Y(\omega)$, will be $Y(\omega) = 0$ for $|\omega| > \omega_0$. Therefore, the Nyquist rate will be $2\,\omega_0$.

(c) The signal $y(t) = (x * z)(t)$ in the frequency domain will be $Y(\omega) = X(\omega)Z(\omega)$. So if $X(\omega) = 0$ for $|\omega| = \frac{\omega_0}{3}$, then $Y(\omega) = 0$ and the Nyquist rate is 0. Otherwise $Y(\omega) = 0$ for $|\omega| > \frac{\omega_0}{3}$. Therefore, the Nyquist rate will be $\frac{2\omega_0}{3}$.

(d) The signal $y(t) = (x * z)(t)$ in the frequency domain will be $Y(\omega) = X(\omega)Z(\omega)$. However, $Z(\omega) = \pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)$ while $X(\omega) = 0$ for $|\omega| > \frac{\omega_0}{2}$, Therefore, $Y(\omega) = 0$ the Nyquist rate will be zero.

Solution 10: A simple communications system II

Many communication systems, for example mobile phones and other wireless devices, send information across free space using electromagnetic waves. To send these electromagnetic waves across long distances, the frequency of the transmitted signal must be very high compared to the frequency of the information signal. An essential technique in designing such communication systems is called *modulation*. During modulation an information signal to be transmitted is embedded, or modulated, onto a higher frequency waveform called carrier. In this problem we analyze a simple communication system that uses the principle of modulation.

Let $x(t)$ be a real-valued signal for which $X(\omega) = 0$ when $|\omega| \geq 2000\pi$. In order to communicate $x(t)$ over free space modulation is performed to produce the transmitted signal $g(t)$, where

$$g(t) = x(t)\cos 2000\pi t.$$

(a) Find the Fourier transform, $G(\omega)$, of the transmitted signal $g(t)$.

Once the signal $g(t)$ is received, it needs to be processed (demodulated) to recover $x(t)$. A proposed demodulation system is illustrated in Figure 5 where $\mathcal{H}$ is an *ideal low pass filter* with a frequency response given by

$$H(\omega) = \begin{cases} b, & |\omega| \leq \omega_c \\ 0, & \text{elswehere.} \end{cases}$$

(b) Find the Fourier transform, $Z(\omega)$, of the input to the low pass filter $z(t)$.

(c) Find the gain b and the cut-off frequency ω_c such that $y(t) = x(t)$.

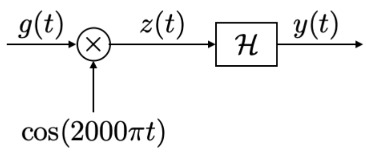

Figure 5: Proposed demodulation system.

Solution: *(a)*

Applying the convolution in frequency property from Appendix 4.A and the Fourier transform pair for cosine form Appendix 4.B we obtain

$$\begin{aligned} G(\omega) &= \frac{1}{2\pi} X(\omega) * \pi(\delta(\omega - 2000\pi) + \delta(\omega + 2000\pi)) \\ &= \frac{1}{2}(X(\omega - 2000\pi) + X(\omega + 2000\pi)). \end{aligned}$$

(b)

Again, applying the convolution in frequency property from Appendix 4.A and the Fourier transform pair for cosine form Appendix 4.B we obtain

$$\begin{aligned} Z(\omega) &= \frac{1}{2\pi} G(\omega) * \pi(\delta(\omega - 2000\pi) + \delta(\omega + 2000\pi)) \\ &= \frac{1}{2}(G(\omega - 2000\pi) + G(\omega + 2000\pi)) \\ &= \frac{1}{4}(X(\omega - 4000\pi) + 2X(\omega) + X(\omega + 4000\pi)) \end{aligned}$$

(c)

By looking at the expression for $Z(\omega)$ we see that $b = 2$ and $\omega_c = 2000\pi$ will produce the desired result. It might be helpful to sketch the Fourier transform of $Z(\omega)$ in terms of the Fourier transform of $X(\omega)$ to see what is going on in the frequency domain.

Solution 11: Impulse-Train sampling

Let $x(t)$ be a signal with Nyquist rate ω_0 and let $x_p(t) = x(t)p(t-1)$, where

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \text{ and } T < \frac{2\pi}{\omega_0}$$

Specify the constraints on the frequency response of a filter that gives $x(t)$ as its output when $x_p(t)$ is the input.

Solution:

From Appendix 4.B, the Fourier transform of $p(t)$ is

$$P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{T}).$$

From Appendix 4.A the Fourier transform of $q(t) = p(t-1)$ is

$$Q(\omega) = e^{-j\omega} P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{T}) e^{-jk\frac{2\pi}{T}}.$$

Since $x_p(t) = x(t)p(t-1)$ we have

$$X_p(\omega) = \frac{1}{2\pi}(X * Q)(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\frac{2\pi}{T}) e^{-jk\frac{2\pi}{T}}.$$

Therefore, $X_p(\omega)$ consists of replicas of $X(\omega)$ shifted by $k\frac{2\pi}{T}$ and added to each other. In order to recover $x(t)$ from $x_p(t)$, we need to isolate one replica of $X(\omega)$ from $X_p(\omega)$ (the one corresponding to the $k=0$ shift). This is accomplished if we multiply $X_p(\omega)$ by

$$H(\omega) = \begin{cases} T, & |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

where $\frac{\omega_0}{2} < \omega_c < \frac{2\pi}{T} - \frac{\omega_0}{2}$.