Problem Set 3

Problem 1: Hard and Easy

(a) A discrete-time sequence $x[n]$ is given as

$$x[n] = x_1[n] - x_1[n+3]$$

where

$$x_1[n] = \left(\frac{1}{2}\right)^n u[n]$$

Find $X(e^{jw})$.

(b) Compute the value of $\int_{-\pi}^{+\pi} X(e^{jw})dw$.

Hint: you could do it the hard way, or the easy way. We recommend finding the easy way!

Problem 2: Frequency response

(a) Consider a continuous-time LTI system with frequency response $H(\omega)$ and real and even impulse response $h(t)$ (that is, $h(t) = h(-t)$). Suppose that we apply an input $x(t) = \sin(\omega_0 t)$ to this system. The resulting output can be shown to be of the form

$$y(t) = A\sin(\omega_0 t)$$

where A is a real number. Find A as a function of $H(\omega)$, and ω_0. For full credit do this without taking a Fourier Transform of $x(t)$.

(b) Consider a continuous-time LTI system with frequency response $H(\omega) = |H(\omega)|e^{j \arg H(\omega)}$ and real impulse response $h(t)$. Suppose that we apply an input $x(t) = \cos(\omega_0 t + \phi_0)$ to this system. The resulting output can be shown to be of the form

$$y(t) = Ax(t - t_0)$$

where A is a nonnegative real number representing an amplitude-scaling and t_0 is a time delay. Find A and t_0 as a function of $H(\omega)$, ω_0, and ϕ_0. For full credit do this without taking a Fourier Transform of $x(t)$.

Problem 3: Ideal filters

(a) A continuous-time ideal low-pass filter has a frequency response

$$H(\omega) = \begin{cases} 1, |\omega| \leq \frac{\pi}{5} \\ 0, \text{ otherwise} \end{cases}$$

Find the output when the filter is applied to each of the following inputs

$$x(t) = \cos\frac{\pi}{3}t, \quad x(t) = 1, \quad x(t) = \text{sinc}\left(\frac{t}{6}\right), \quad x(t) = \text{sinc}\left(\frac{t}{4}\right). \tag{1}$$

(b) A discrete-time ideal low-pass filter has a frequency response

$$H(e^{j\omega}) = \begin{cases} 1, |\omega| \leq \frac{\pi}{5} \\ 0, \text{ otherwise} \end{cases}$$

on the interval $-\pi < \omega \leq \pi$. Find the output when the filter is applied to each of the following inputs

$$x[n] = \cos\frac{13\pi}{6}n, \quad x[n] = (-1)^n, \quad x[n] = \delta[n]. \tag{2}$$

Problem 4: Composition of systems

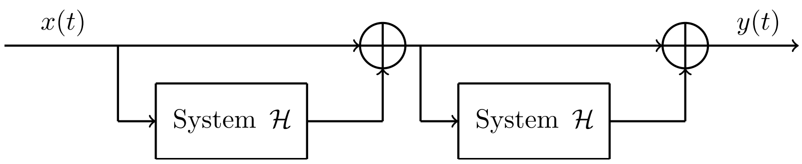

Figure 1: A composed system.

In this problem, we study the system composition illustrated in Figure 1 with input $x(t)$ and output $y(t)$, where we **assume** that the system $\mathcal{H}$ is known to be LTI and stable.

(a) Prove that the composed system shown in the figure is stable.

(b) Give the frequency response of the overall system in Figure 1 with input $x(t)$ and output $y(t)$ in terms of the frequency response $H(\omega)$ of the component system $\mathcal{H}$.

(c) For the special case where the system $\mathcal{H}$ is the LTI system with impulse response $h(t) = e^{-|t|}$, give an explicit formula for the frequency response of the overall system (expressed as a ratio of two polynomials in ω).

Problem 5: Step response (Review)

During the course we have seen as the impulse response completely characterizes an LTI system. Unfortunately, it is rather difficult to produce an infinite amplitude pulse with infinitesimal duration in practice. In this problem we define another type of response called the *step response* which is often used to characterize an LTI system. It is defined, in continuous time, as:

$$s(t) = u(t) * h(t)$$

That is, $s(t)$ characterizes how an LTI system reacts to the signal $u(t)$.

(a) An LTI system is known to have an impulse response

$$h(t) = e^{-\alpha t} u(t - t_0)$$

where $\alpha, t_0 > 0$. Find its step response.

(b) Use the convolution integral to show

$$h(t) = \frac{ds(t)}{dt} = s'(t).$$

(c) Recall that input-output relationship of a stable LTI system could be related by the equation

$$Y(\omega) = H(\omega)X(\omega)$$

where $Y(\omega)$ is the Fourier transform for the output signal $y(t)$, $X\omega)$ is the Fourier transform for the input signal $x(t)$, and $H(\omega)$ is the Fourier transform for the impulse response $h(t)$.

Assume that the Fourier Transform of $s(t)$ of an LTI system exists. Find the same input-output relationship in the frequency domain in terms of $S(\omega)$. That is, you need to derive the formula that relates $X(\omega)$ to $Y(\omega)$ in terms of $S(\omega)$.

Problem 6: A simple communication system

Many communication systems, for example mobile phones and other wireless devices, send information across free space using electromagnetic waves. To send these electromagnetic waves across long distances, the frequency of the transmitted signal must be very high compared to the frequency of the information signal. An essential technique in designing such communication systems is called *modulation.* During modulation an information signal to be transmitted is embedded, or modulated, onto a higher frequency waveform called carrier. In this problem we analyze a simple communication system that uses the principle of modulation.

Let $x(t)$ be a real-valued signal for which $X(\omega) = 0$ when $|\omega| \geq 2000\pi$. In order to communicate $x(t)$ over free space modulation is performed to produce the transmitted signal $g(t)$, where

$$g(t) = x(t) \cos 2000\pi t.$$

(a) Find the Fourier transform, $G(\omega)$, of the transmitted signal $g(t)$.

Once the signal $g(t)$ is received, it needs to be processed (demodulated) to recover $x(t)$. A proposed demodulation system is illustrated in Figure 3 where $\mathcal{H}$ is an *ideal low pass filter* with a frequency response given by

$$H(\omega) = \begin{cases} b, & |\omega| \leq \omega_c \\ 0, & \text{elswehere.} \end{cases}$$

(b) Find the Fourier transform, $Z(\omega)$, of the input to the low pass filter $z(t)$.

(c) Find the gain b and the cut-off frequency ω_c such that $y(t) = x(t)$.

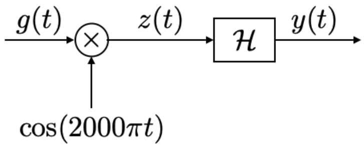

Figure 2: Proposed demodulation system.

Problem 7: Sampling sinusoids

(a) A sampling system samples continuous-time signals with frequency $\omega_s = 1000\pi$. According to the sampling theorem, which of the following signals could be reconstructed exactly if sampled by this system:

i. $x(t) = \cos 200\pi t$, *ii.* $x(t) = \sin 450\pi t$, *iii.* $x(t) = \sin 2500\pi t$, *iv.* $x(t) = \cos 200\pi t + \sin 800\pi t$.

(b) A sampling system samples continuous-time signals with sampling interval $T = 0.5 \times 10^{-3}$. According to the sampling theorem, which of the following signals could be reconstructed exactly if sampled by this system:

i. $x(t) = \cos 1000\pi t$, *ii.* $x(t) = \sin 2500\pi t$, *iii.* $x(t) = \cos 500\pi t + \sin 300\pi t$, *iv.* $x(t) = \sin 1500\pi t$.

Problem 8: Sampling sinusoids - Part 2

A mystery signal $x(t)$ is sampled with frequency $\omega_s = 1000\pi$ using impulse-train sampling, and then reconstructed with a low-pass filter with cut-off frequency $\omega_c = 500\pi$. The reconstructed signal is

$$x_r(t) = \cos 200\pi t.$$

We do not know anything else about $x(t)$. Which of the following signals could be $x(t)$?

i. $x(t) = \cos 300\pi t$, *ii.* $x(t) = \cos 200\pi t$, *iii.* $x(t) = \cos 1200\pi t$, *iv.* $x(t) = \cos 800\pi t$.

Make sure to justify your answers for full credit.

Problem 9: Nyquist rate

Let $x(t)$ be a signal with Nyquist rate ω_0. Determine the Nyquist rate for each of the following signals:

(a) $x(t) - x(t-1)$

(b) $x(t)x(t-1)$

(c) $(x * z)(t)$ where $z(t) = \sin \frac{\omega_0}{3} t$
Hint: A complete answer should include two possibilities.

(d) $(x * z)(t)$ where $z(t) = \cos \omega_0 t$

Problem 10: A simple communications system II

Many communication systems, for example mobile phones and other wireless devices, send information across free space using electromagnetic waves. To send these electromagnetic waves across long distances, the frequency of the transmitted signal must be very high compared to the frequency of the information signal. An essential technique in designing such communication systems is called *modulation*. During modulation an information signal to be transmitted is embedded, or modulated, onto a higher frequency waveform called carrier. In this problem we analyze a simple communication system that uses the principle of modulation.

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(a) Find the Fourier transform, $G(\omega)$, of the transmitted signal $g(t)$.

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$$H(\omega) = \begin{cases} b, & |\omega| \leq \omega_c \\ 0, & \text{elswehere.} \end{cases}$$

(b) Find the Fourier transform, $Z(\omega)$, of the input to the low pass filter $z(t)$.

(c) Find the gain b and the cut-off frequency ω_c such that $y(t) = x(t)$.

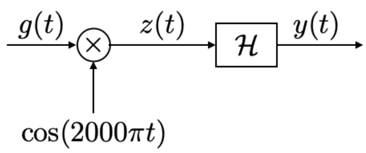

Figure 3: Proposed demodulation system.

Problem 11: Impulse-Train sampling

Let $x(t)$ be a signal with Nyquist rate ω_0 and let $x_p(t) = x(t)p(t-1)$, where

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \text{ and } T < \frac{2\pi}{\omega_0}$$

Specify the constraints on the frequency response of a filter that gives $x(t)$ as its output when $x_p(t)$ is the input.