Fundamentals of electrical circuits and systems I
Prof. Jean-Philippe Thiran September 30, 2025

Solution Set 2

Solution 1: Calculation of Fourier transform

In this problem we will find the Fourier Transform of the signal

$$z(t) = \begin{cases} \cos \omega_0 t, & \text{if } |t| \leq a_0 \\ 0, & \text{otherwise} \end{cases}$$

using two different methods.

(a) Use Fourier Pairs table in Appendix 4.B to find the Fourier transform of signals $x(t) = \cos \omega_0 t$ and

$$y(t) = \begin{cases} 1, & \text{if } |t| \leq a_0 \\ 0, & \text{otherwise.} \end{cases}$$

Notice that we could write $z(t) = x(t) \cdot y(t)$. Find $Z(\omega)$ by using this insight and tables in Appendix 4.A and 4.B.

(b) Notice also that the signal $z(t)$ is absolutely integrable and its Fourier transform is well-defined. Find $Z(\omega)$ directly by using the Fourier transform equation.

(c) Confirm that part (a) is indeed the same as part (b).

Solution

(a) From Appendix 4.B

$$X(\omega) = \pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$$

and

$$Y(\omega) = 2a_0 \text{sinc}\left(\frac{a_0}{\pi}\omega\right)$$

where $Y(\omega)$ is obtained by applying the Box function Fourier pair to $\sqrt{2a_0}b(t)$ with $t_0 = 2a_0$.

From the convolution in frequency property in Appendix 4.A we then have

$$Z(\omega) = \frac{1}{2\pi}(X * Y)(\omega) = a_0 \text{sinc}\left(\frac{a_0}{\pi}(\omega - \omega_0)\right) + a_0 \text{sinc}\left(\frac{a_0}{\pi}(\omega + \omega_0)\right)$$

(b) By Fourier transform formula we have:

$$
\begin{aligned}
Z(\omega) &= \int_{-\infty}^{+\infty} z(t)e^{-j\omega t}dt \\
&= \int_{-a_0}^{a_0} \cos(\omega_0 t)e^{-j\omega t}dt \\
&= \int_{-a_0}^{a_0} \left(\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}\right) e^{-j\omega t}dt \\
&= \frac{1}{2}\int_{-a_0}^{a_0} \left(e^{-j(\omega-\omega_0)t}+e^{-j(\omega+\omega_0)t}\right) dt \\
&= \frac{1}{2}\frac{1}{j(\omega-\omega_0)}(e^{j(\omega-\omega_0)a_0}-e^{-j(\omega-\omega_0)a_0})+\frac{1}{2}\frac{1}{j(\omega+\omega_0)}(e^{j(\omega+\omega_0)a_0}-e^{-j(\omega+\omega_0)a_0}) \\
&= \frac{\sin\left(a_0(\omega-\omega_0)\right)}{\omega-\omega_0}+\frac{\sin\left(a_0(\omega+\omega_0)\right)}{\omega+\omega_0} \\
&= a_0\mathrm{sinc}\left(\frac{a_0}{\pi}(\omega-\omega_0)\right)+a_0\mathrm{sinc}\left(\frac{a_0}{\pi}(\omega+\omega_0)\right).
\end{aligned}
$$

(c) Indeed, parts (a) and (b) are the same once we represented the answer as a sum of sincs.

Solution 2: Properties of Fourier transform

Appendix 4.A in the lecture notes summarizes the properties of continuous-time Fourier transform.

(a) Assume that $x_1(t) \circ\!\!-\!\!\bullet X_1(\omega)$ and $x_2(t) \circ\!\!-\!\!\bullet X_2(\omega)$. Express the Fourier transform of the signal

$$y(t) \quad = \quad x_1(t-3)+x_2\left(\frac{t}{2}\right)$$

in terms of $X_1(\omega)$ and $X_2(\omega)$ using Appendix 4.A in the lecture notes.

(b) Assume that $x(t)\circ\!\!-\!\!\bullet X(\omega)$, $y(t)\circ\!\!-\!\!\bullet Y(\omega)$ and $z(t)\circ\!\!-\!\!\bullet Z(\omega)$. For two absolutely integrable real-valued functions $x(t)$ and $y(t)$, express the continuous-time Fourier transform $Z(\omega)$ of the signal

$$z(t) \quad = \quad \int_{-\infty}^{\infty} x(\tau)y(t+\tau)d\tau$$

in terms of $X(\omega)$ and $Y(\omega)$.

(c) Assume that $x(t) \circ\!\!-\!\!\bullet X(\omega)$. Express the Fourier transform of the signal

$$y(t) \quad = \quad x(2t-5)$$

in terms of $X(\omega)$.

(d) Assume that $x_1(t)\circ\!\!-\!\!\bullet X_1(\omega)$ and $x_2(t)\circ\!\!-\!\!\bullet X_2(\omega)$. Find the inverse Fourier transform of the function

$$Y(\omega) \quad = \quad 2X_1^*(\omega-5)+2\frac{d}{d\omega}X_2(\frac{\omega}{2})$$

in terms of $x_1(t)$ and $x_2(t)$ using Appendix 4.A in the lecture notes.

Solution

(a) Let us start with the first term $x_1(t-3)$:

1. time-shifting: $x_1(t-3) \circ\!\!-\!\!\bullet e^{-j3\omega}X_1(\omega)$.

Next, we consider the second term $x_2(\frac{t}{2})$:

1. time scaling: $x_2(\frac{t}{2}) \circ\!\!-\!\!\bullet\, 2X_2\,(2\omega)$,

Finally, applying the property of linearity, we have

$$Y(\omega) \quad = \quad e^{-j3\omega}X_1(\omega) + 2X_2\,(2\omega)\,.$$

(b) We start by defining $\mu = -\tau$ and we obtain

$$z(t) \quad = \quad -\int_{\infty}^{-\infty} x(-\mu)y(t-\mu)d\mu \tag{1}$$

Now, if we define $\tilde{x}(t) = x(-t)$, we obtain

$$z(t) \quad = \quad \int_{-\infty}^{\infty} \tilde{x}(\mu)y(t-\mu)d\mu \tag{2}$$

which is a standard convolution, hence, from the convolution property,

$$Z(\omega) \quad = \quad \tilde{X}(\omega)Y(\omega),$$

and from the Fourier properties, if $\tilde{x}(t) = x(-t)$, then $\tilde{X}(\omega) = X(-\omega)$, hence, the final answer is

$$Z(\omega) \quad = \quad X(-\omega)Y(\omega).$$

(c) Remember to shift first, and scale second. Consider the term $x(2t-5)$:

1. time-shifting: $x(t-5) \circ\!\!-\!\!\bullet\, e^{-j\omega 5}X(\omega)$,
2. time-scaling: $x(2t-5) \circ\!\!-\!\!\bullet\, \frac{1}{2}e^{-j\omega 5/2}X(\frac{\omega}{2})$.

(d) Using the property of conjugation and time-reversal, it can be shown that $x^*(-t) \circ\!\!-\!\!\bullet\, X^*(\omega)$. Let us start with the first term $X_1^*(\omega-5)$:

1. conjugation (frequency): $x_1^*(-t) \circ\!\!-\!\!\bullet\, X_1^*(\omega)$,
2. frequency-shifting: $e^{j5t}x_1^*(-t) \circ\!\!-\!\!\bullet\, X_1^*(\omega-5)$.

Next, we consider the second term $\frac{d}{d\omega}X_2(\frac{\omega}{2})$:

1. frequency scaling: $2x_2\,(2t) \circ\!\!-\!\!\bullet\, X_2(\frac{\omega}{2})$,
2. differentiation in frequency: $-2jtx_2\,(2t) \circ\!\!-\!\!\bullet\, \frac{d}{d\omega}X_2(\frac{\omega}{2})$.

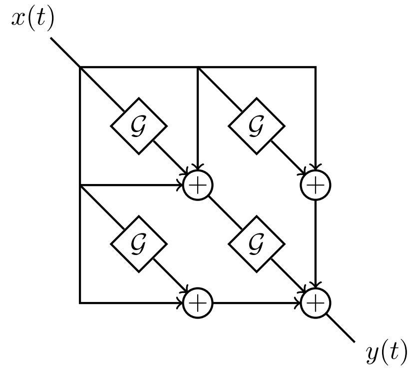
Finally, applying the property of linearity, we have

$$y(t) \quad = \quad 2e^{j5t}x_1^*(-t) - 4jtx_2\,(2t)\,.$$

Parsing this problem into subproblems, we define $Z_1(\omega) = X_1^*(\omega)$ and $Z_2(\omega) = X_2(\frac{\omega}{2})$, thus $Y(\omega) = 2Z_1(\omega-5)+2\frac{d}{d\omega}Z_2(\omega)$. By using the properties listed above, in time domain we would have $z_1(t) = x_1^*(-t)$ and $z_2(t) = 2x_2(2t)$, thus $y(t) = 2e^{j5t}z_1(t) + -2jtz_2(t)$. By substituting signal $z_1(t)$ and $z_2(t)$, then the expression will be in terms of $x_1(t)$ and $x_2(t)$.

Solution 3: Composition of Systems

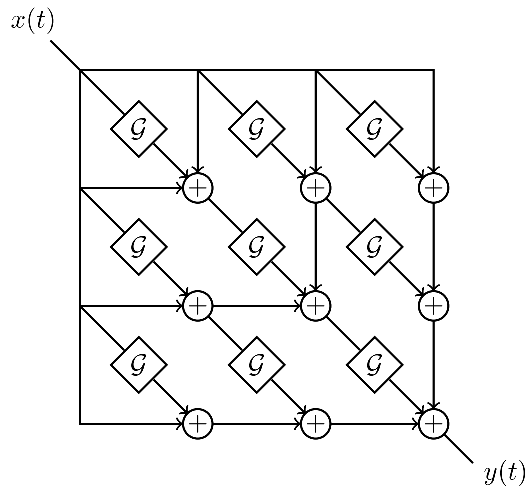
Assume that the system $\mathcal{G}$ is stable and linear time-invariant with impulse response $g(t)$. We combine many systems $\mathcal{G}$ in a systematic manner as follows.



(a) Find the overall impulse response of the system with input $x(t)$ and output $y(t)$ in terms of $g(t)$.

(b) (Optional Additional Challenge: The system in the figure can be interpreted as a 2×2 composition, and we can naturally extend it to a 3×3 composition, and further to a general $D \times D$ composition. Find the corresponding overall impulse response $h_D(t)$.

Hint: Start with $D = 3$ to observe the pattern as depicted in the figure below. If $h_D(t)$ is the overall impulse response of a $D \times D$ concatenation, then try to write $h_D(t)$ in terms of $h_{D-1}(t)$.)

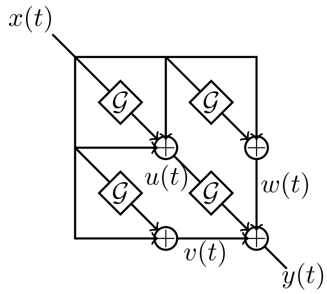


Solution

(a) In this type of problems we introduce auxiliary signals. A choice of such signals is indicated in the figure below.

From the figure below we can relate $u(t)$, $v(t)$ and $w(t)$ with $x(t)$.

Thus, we have $u(t) = 2x(t) + (g * x)(t) = (2\delta(t) + g(t)) * x(t)$, $v(t) = w(t) = (\delta(t) + g(t)) * x(t)$. Therefore



the overall impulse response is

$$\begin{aligned}
y(t) &= w(t) + v(t) + (u * g)(t) \\
&= 2(\delta(t) + g(t)) * x(t) + (2\delta(t) + g(t)) * g(t) * x(t) \\
&= x(t) * [2(\delta(t) + g(t)) + (2\delta(t) + g(t)) * g(t)] \\
&= x(t) * \underbrace{(2\delta(t) + 4g(t) + g(t) * g(t))}_{h(t)}
\end{aligned}$$

(b) First we start with a 1×1 system which has the impulse response $h_1(t) = 2\delta(t) + g(t)$. Observe that the transition from the impulse responses of the $D \times D$ system to the impulse response of $(D-1) \times (D-1)$ system is $h_D(t) = h_{D-1}(t) * g(t) + 2(\delta(t) + g(t))^{*(D-1)}$ where we use the notation $z(t)^{*(D)}$ to denote D times of convolution between $z(t)$.

Apply the recursion multiple-times we have

$$\begin{aligned}
h_D(t) &= h_{D-1}(t) * g(t) + 2(\delta(t) + g(t))^{*D-1} \\
&= 2(\delta(t) + g(t))^{*D} - g(t)^{*D}.
\end{aligned}$$

To see why this is, it might be easier to think about in the frequency domain where convolution becomes multiplication. The strategy here would be to find the frequency response of the overall system, and invert it to get the impulse response. Thus, applying the recursion multiple-times we have

$$\begin{aligned}
H_D(\omega) &= H_{D-1}(\omega)G(\omega) + 2(1 + G(\omega))^{D-1} \\
&= (H_{D-2}(\omega)G(\omega) + 2(1 + G(\omega))^{D-2})G(\omega) + 2(1 + G(\omega))^{D-1} \\
&= H_{D-2}(\omega)G(\omega)^2 + 2G(\omega)(1 + G(\omega))^{D-2} + 2(1 + G(\omega))^{D-1} \\
&= H_{D-3}(\omega)G(\omega)^3 + 2G(\omega)^2(1 + G(\omega))^{D-3} + 2G(\omega)(1 + G(\omega))^{D-2} + 2(1 + G(\omega))^{D-1} \\
&\vdots \quad \vdots \\
&= H_1(\omega)G(\omega)^{D-1} + 2G(\omega)^{D-2}(1 + G(\omega)) + 2G(\omega)^{D-3}(1 + G(\omega))^2 + \cdots + 2(1 + G(\omega))^{D-1} \\
&= (2 + G(\omega))G(\omega)^{D-1} + 2G(\omega)^{D-2}(1 + G(\omega)) + 2G(\omega)^{D-3}(1 + G(\omega))^2 + \cdots + 2(1 + G(\omega))^{D-1} \\
&= G(\omega)^D + 2\left[G(\omega)^{D-1} + G(\omega)^{D-2}(1 + G(\omega)) + \cdots + G(\omega)(1 + G(\omega))^{D-2} + (1 + G(\omega))^{D-1}\right] \\
&= G(\omega)^D + 2\frac{(1 + G(\omega))^D - G(\omega)^D}{1 + G(\omega) - G(\omega)} \\
&= G(\omega)^D + 2(1 + G(\omega))^D - 2G(\omega)^D \\
&= 2(1 + G(\omega))^D - G(\omega)^D.
\end{aligned}$$

Solution 4: Directly computing discrete-time Fourier Transform

Use the discrete-time Fourier transform equation to find the Fourier transform of the following signals:

(a) $x[n] = \frac{j}{2}\delta[n+1] - \frac{j}{2}\delta[n-1]$,

(b) $x[n] = -\alpha^n u[-n-1]$ where $|\alpha| > 1$.

Use the discrete-time inverse Fourier transform equation to find the inverse Fourier transform of the following signal:

(c) $X(\omega) = \sum_{k=-\infty}^{\infty} \left(2\pi\delta(\omega - 2\pi k) + \pi\delta(\omega - \frac{\pi}{3} - 2\pi k) + \pi\delta(\omega + \frac{\pi}{3} - 2\pi k)\right)$.

Solution:

(a)

$$\begin{aligned}
X(\omega) &= \sum_{n=-\infty}^{\infty} \frac{j}{2}\left(\delta[n+1] - \delta[n-1]\right) e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \frac{j}{2}\delta[n+1] e^{-j\omega n} - \sum_{n=-\infty}^{\infty} \frac{j}{2}\delta[n-1] e^{-j\omega n} \\
&= \frac{j}{2}e^{j\omega} - \frac{j}{2}e^{-j\omega} = -\sin(\omega)
\end{aligned}$$

(b)

$$\begin{aligned}
X(\omega) &= \sum_{n=-\infty}^{\infty} -\alpha^n u[-n-1] e^{-j\omega n} \\
&= \sum_{n=-\infty}^{-1} -\alpha^n e^{-j\omega n} \\
&= \sum_{n=1}^{\infty} -\left(\frac{1}{\alpha}e^{j\omega}\right)^n \\
&= -\frac{1}{1 - \frac{1}{\alpha}e^{j\omega}} + 1 \\
&= \frac{\alpha e^{-j\omega}}{1 - \alpha e^{-j\omega}} + 1 \\
&= \frac{1}{1 - \alpha e^{-j\omega}}
\end{aligned}$$

(c)

$$\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} \left(2\pi\delta(\omega - 2\pi k) + \pi\delta(\omega - \frac{\pi}{3} - 2\pi k) + \pi\delta(\omega + \frac{\pi}{3} - 2\pi k)\right) e^{j\omega n} d\omega \\
&\overset{(*)}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(2\pi\delta(\omega) + \pi\delta(\omega - \frac{\pi}{3}) + \pi\delta(\omega + \frac{\pi}{3})\right) e^{j\omega n} d\omega \\
&= 1 + \frac{1}{2}e^{-j\frac{\pi}{3}n} + \frac{1}{2}e^{j\frac{\pi}{3}n} \\
&= 1 + \cos\left(\frac{\pi}{3}n\right)
\end{aligned}$$

where step $(*)$ is consequence of the fact that all the dirac deltas for $k \neq 0$ are not present on the interval $[-\pi\,,\,\pi)$. As an example for $k=1$, the valid dirac deltas are $\delta(\omega-2\pi)$, $\delta(\omega-\frac{\pi}{2}-2\pi)$ and $\delta(\omega+\frac{\pi}{2}-2\pi)$, where all of them are zero in the interval $[-\pi\,,\,\pi)$.

Solution 5: LTI Systems : input/output relationship

One of the main results of this class is that (stable) LTI systems can either be tackled in the time domain or in the frequency (Fourier) domain. In the time domain, every LTI system can be characterized by its impulse response, and the input-output relationship can be written as $y(t)=(h*x)(t)$. In the frequency domain, every stable LTI system can be characterized by its frequency response, and the input-output relationship can be written as $Y(\omega)=H(\omega)X(\omega)$. Depending on the task and the signals at hand, one of the two representations can be substantially simpler than the other. As you become more proficient in the use of these fundamental tools, you will start to develop an intuition for which one to select, but in general, it definitely pays off to briefly consider both approaches, and then decide which one to select. In the following, feel free to make use of the formulas in Appendices 4.A and 4.B in the lecture notes. Find $y(t)$.

(a) $h(t)=e^{-at}u(t)$, $x(t)=e^{-bt}u(t)$, where $a>b>0$.

(b) $h(t)=\frac{8}{\pi}\mathrm{sinc}(\frac{8}{\pi}(t-2))$, and $x(t)=\frac{1}{\pi}\left(\mathrm{sinc}(\frac{1}{\pi}t)\right)^2$.

Solution

(a) In the **time domain,** we can proceed as follows:

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)e^{-b(t-\tau)}u(t-\tau)d\tau \\
&= \int_{0}^{\infty} e^{-a\tau}e^{-b(t-\tau)}u(t-\tau)d\tau \\
&= \left\{\int_{0}^{t} e^{-a\tau}e^{-b(t-\tau)}d\tau\right\}u(t) \\
&= \left\{e^{-bt}\int_{0}^{t} e^{-(a-b)\tau}d\tau\right\}u(t)
\end{aligned}$$

Alternatively, we can tackle the problem in the **frequency domain,** as follows: First, we have

$$\begin{aligned}
H(\omega) &= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}\,dt \\
&= \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt \\
&= \frac{1}{a+j\omega}.
\end{aligned}$$

Similarly, we have $X(\omega)=\frac{1}{b+j\omega}$. Thus, the output in the frequency domain can be formed as

$$\begin{aligned}
Y(\omega) &= \frac{1}{a+j\omega}\frac{1}{b+j\omega} \\
&= \frac{1}{a-b}\left(\frac{1}{b+j\omega}-\frac{1}{a+j\omega}\right).
\end{aligned}$$

Finally, using $e^{-at}u(t) \circ\!\!-\!\!\bullet \frac{1}{a+j\omega}$ for all $a>0$, the output in the time domain is given by

$$y(t) = \frac{e^{-bt}-e^{-at}}{a-b}u(t).$$

(b) First, let us define

$$g(t) = h(t+2) = \frac{8}{\pi}\mathrm{sinc}(\frac{8}{\pi}t) \tag{3}$$

From the Appendix 5.B of the lecture notes, second-to-last row, we have

$$G(\omega) = \begin{cases} 1, & |\omega| \le 8 \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Using the shift in time property, $G(\omega) = e^{2j\omega}H(\omega)$, thus we have

$$H(\omega) = e^{-2j\omega}G(\omega) = \begin{cases} e^{-2j\omega}, & |\omega| \le 8 \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Now, to find the Fourier transform of $x(t)$ is a little more tricky. Let us define

$$z(t) = \sqrt{\frac{1}{\pi}}\mathrm{sinc}(\frac{1}{\pi}t), \tag{6}$$

so that $x(t) = z^2(t)$. Clearly, again from the Appendix 5.B of the lecture notes, second-to-last row, we have

$$Z(\omega) = \begin{cases} \sqrt{\pi}, & |\omega| \le 1 \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Now, how to find $X(\omega)$ from $Z(\omega)$? One idea is to realize that we can use the property "Convolution in Frequency" from Appendix 5.A of the lecture notes. Using this, we find

$$X(\omega) = \frac{1}{2\pi}(Z * Z)(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(\nu)Z(\omega-\nu)d\nu. \tag{8}$$

You can directly solve this integral. Alternatively (my preferred route...), you can observe that $Z(\omega)$ is just a box function (in the variable ω). That is, we are convolving a box function with itself. This is something we studied when we did flip-and-drag, and hopefully you all recall that this gives a nice triangle shape. Moreover, since the box is centered at the origin, the triangle is also centered at the origin. This means that all that is left to compute is:

1. The width of the triangle. Since $Z(\omega)$ lives between $-1 \le \omega \le 1$, we know that the triangle representing $X(\omega)$ lives between $-2 \le \omega \le 2$. Done.
2. The height of the triangle, i.e., the amplitude at $\omega = 0$. As you recall, this is simply the integral of $Z^2(\omega)$, which is 2π. But since here, we are not just doing convolution, but we also need to normalize (divide) by 2π, we conclude that $X(\omega = 0) = 1$.

In summary, we have thus found that

$$X(\omega) = \begin{cases} 1 + \omega/2, & -2 \le \omega \le 0, \\ 1 - \omega/2, & 0 \le \omega \le 2, \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

Moreover, we have

$$Y(\omega) = H(\omega)X(\omega) = e^{-2j\omega}X(\omega), \tag{10}$$

where the last step is key and needs to be understood carefully! Specifically, it holds because the interval where $X(\omega)$ is non-zero is *inside* the interval where $H(\omega)$ is non-zero. Therefore, we automatically keep track of the interval where $H(\omega)$ is zero, and do not need to keep this around. This fact *considerably* simplifies our manipulations, as follows: Using the shift in time property, we conclude that

$$y(t) = x(t-2). \tag{11}$$

Solution 6: Discrete-time Fourier transform properties

Given that $x[n]$ has Fourier transform $X(\omega)$, express the Fourier transform of the following signals in terms of $X(\omega)$:

(a) $y[n] = (n - n_0)x[n - n_0]$,

(b) $y[n] = x[n]\cos(\omega_0 n)$,

(c) $y[n] = (-1)^n x[n]$,

Solution:

(a) There are two ways to solve this problem, both involve an application of *Shift in Time* and *Differentiation in Frequency* properties.

First, let us write

$$\tilde{y}[n] = y[n + n_0] = nx[n].$$

Then, by the Shift in Time property

$$\tilde{Y}(\omega) = e^{j\omega n_0} Y(\omega)$$

and by the Differentiation in Frequency property

$$\tilde{Y}(\omega) = j\frac{d}{d\omega}X(\omega).$$

Putting this together we obtain

$$Y(\omega) = je^{-j\omega n_0}\frac{d}{d\omega}X(\omega).$$

Alternatively, let us write

$$\begin{aligned} y[n] &= nx[n - n_0] - n_0 x[n - n_0] \\ &= n\tilde{x}[n] - n_0\tilde{x}[n] \end{aligned}$$

where $\tilde{x}[n] = x[n - n_0]$. Then, by the Shift in Time property

$$\tilde{X}(\omega) = e^{-j\omega n_0} X(\omega)$$

and by the Linearity property and by Differentiation in Frequency property

$$\begin{aligned} Y(\omega) &= j\frac{d}{d\omega}\tilde{X}(\omega) - n_0\tilde{X}(\omega) \\ &= j\frac{d}{d\omega}\left(e^{-j\omega n_0}X(\omega)\right) - n_0 e^{-j\omega n_0}X(\omega) \end{aligned}$$

You may wish to simplify this expression further to confirm that the second approach does give the same answer as the first approach.

(b) The simplest way to solve this problem is to apply Euler's formula to the $\cos(\omega_0 n)$ and then to use *Shift in Frequency* property.

First, apply Euler's formula:

$$y[n] = x[n]\cos(\omega_0 n) = x[n]\left(\frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}\right) = \frac{1}{2}e^{j\omega_0 n}x[n] + \frac{1}{2}e^{-j\omega_0 n}x[n].$$

Then, by the Linearity property and the Shift in Frequency property

$$Y(\omega) = \frac{1}{2} X(\omega - \omega_0) + \frac{1}{2} X(\omega + \omega_0).$$

(c) We can write $(-1)^n = e^{j\pi n}$ and $y[n] = e^{j\pi n} x[n]$. Then, by the Shift in Frequency property

$$Y(\omega) = X(\omega - \pi).$$

Alternatively, observe that we can write $(-1)^n = \cos(\pi n)$ and $y[n] = \cos(\pi n) x[n]$. But, this is just a special case of part (b) with $\omega_0 = \pi$ and

$$Y(\omega) = \frac{1}{2} X(\omega - \pi) + \frac{1}{2} X(\omega + \pi).$$

Remembering that the discrete-time Fourier transform is 2π periodic we can also write this as

$$Y(\omega) = X(\omega - \pi).$$

Solution 7: System characterization (discrete)

Suppose we are given one input/output pair of an unknown system. Namely, if the input signal is

$$x[n] = \left(\frac{1}{3}\right)^n u[n] - \frac{4}{3}\left(\frac{1}{3}\right)^{n-1} u[n-2],$$

then we observe the following output:

$$y[n] = \left(\frac{4}{5}\right)^n u[n].$$

(a) Find the impulse response and the frequency response of a discrete-time LTI system that gives this output to the mentioned input.

Hint: $H(\omega) = \frac{A}{1-\frac{2}{3}e^{-j\omega}} + \frac{B}{1+\frac{2}{3}e^{-j\omega}} + \frac{C}{1-\frac{4}{5}e^{-j\omega}}$, where $A = -\frac{5}{4}$, $B = \frac{15}{44}$ and $C = \frac{21}{11}$. The hint is given to help you out with the calculation. It is your duty to prove also the hint.

(b) Find a difference equation that relates $x[n]$ and $y[n]$.

Solution:

(a) We will exploit the relationship $H(\omega) = \frac{Y(\omega)}{X(\omega)}$.

$$Y(\omega) = \frac{1}{1 - \frac{4}{5}e^{-j\omega}}$$

For $x[n]$ we will exploit the delay property as mentioned in Appendix 5.C from the Lecture Notes. To that end, define:

$$\begin{aligned} x[n] &= \underbrace{\left(\frac{1}{3}\right)^n u[n]}_{g[n]} - \frac{4}{3}\left(\frac{1}{3}\right)^{n-1} u[n-2] \\ &= g[n] - \frac{4}{9} g[n-2]. \end{aligned}$$

Hence

$$X(\omega) = (1 - \frac{4}{9}e^{-j2\omega})G(\omega)$$
$$= \frac{1 - \frac{4}{9}e^{-j2\omega}}{1 - \frac{1}{3}e^{-j\omega}}.$$

Therefore, we find

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1 - \frac{1}{3}e^{-j\omega}}{(1 - \frac{4}{9}e^{-j2\omega})(1 - \frac{4}{5}e^{-j\omega})}.$$

We try partial fraction expansion to find a simpler expression to which - hopefully - we will know the inverse Fourier transform:

$$\frac{1 - \frac{1}{3}e^{-j\omega}}{(1 - \frac{4}{9}e^{-j2\omega})(1 - \frac{4}{5}e^{-j\omega})} = \frac{A}{1 - \frac{2}{3}e^{-j\omega}} + \frac{B}{1 + \frac{2}{3}e^{-j\omega}} + \frac{C}{1 - \frac{4}{5}e^{-j\omega}}.$$

For this to hold, we must have that -by cross-multiplication-

$$A(1 + \frac{2}{3}e^{-j\omega})(1 - \frac{4}{5}e^{-j\omega}) + B(1 - \frac{2}{3}e^{-j\omega})(1 - \frac{4}{5}e^{-j\omega}) + C(1 - \frac{4}{9}e^{-2j\omega}) = 1 - \frac{1}{3}e^{-j\omega}$$

or equivalently,

$$A(1 - \frac{2}{15}e^{-j\omega} - \frac{8}{15}e^{-j2\omega}) + B(1 - \frac{22}{15}e^{-j\omega} + \frac{8}{15}e^{-j2\omega}) + C(1 - \frac{4}{9}e^{-2j\omega}) = 1 - \frac{1}{3}e^{-j\omega}$$

which results in the following system of equations

$$\begin{aligned} A + B + C &= 1 \\ -\frac{2}{15}A - \frac{22}{15}B &= -\frac{1}{3} \\ -\frac{4}{3}(\frac{2}{5}A - \frac{2}{5}B + \frac{1}{3}C) &= 0. \end{aligned}$$

From this we find $A = -\frac{5}{4}$, $B = \frac{15}{44}$ and $C = \frac{21}{11}$. This gives us the simple inverse Fourier pair:

$$\begin{aligned} H(\omega) &= \frac{-\frac{5}{4}}{1 - \frac{2}{3}e^{-j\omega}} + \frac{\frac{15}{44}}{1 + \frac{2}{3}e^{-j\omega}} + \frac{\frac{21}{11}}{1 - \frac{4}{5}e^{-j\omega}} \Longleftrightarrow \\ h[n] &= -\frac{5}{4}\left(\frac{2}{3}\right)^n u[n] + \frac{15}{44}\left(-\frac{2}{3}\right)^n u[n] + \frac{21}{11}\left(\frac{4}{5}\right)^n u[n]. \end{aligned}$$

(b) From Part *(a)*, we know

$$\frac{Y(\omega)}{X(\omega)} = \frac{1 - \frac{1}{3}e^{-j\omega}}{(1 - \frac{4}{9}e^{-j2\omega})(1 - \frac{4}{5}e^{-j\omega})}.$$

By cross-multiplication:

$$(1 - \frac{4}{9}e^{-j2\omega})(1 - \frac{4}{5}e^{-j\omega})Y\left(e^{j\omega}\right) = (1 - \frac{1}{3}e^{-j\omega})X\left(e^{j\omega}\right),$$

and by taking the inverse Fourier transform, we find:

$$y[n] - \frac{4}{5}y[n-1] - \frac{4}{9}y[n-2] + \frac{16}{45}y[n-3] = x[n] - \frac{1}{3}x[n-1].$$