Problem Set 2

Problem 1: Calculation of Fourier transform

In this problem we will find the Fourier Transform of the signal

$$z(t) = \begin{cases} \cos \omega_0 t, & \text{if } |t| \leq a_0 \\ 0, & \text{otherwise} \end{cases}$$

using two different methods.

(a) Use Fourier Pairs table in Appendix 4.B to find the Fourier transform of signals $x(t) = \cos \omega_0 t$ and

$$y(t) = \begin{cases} 1, & \text{if } |t| \leq a_0 \\ 0, & \text{otherwise.} \end{cases}$$

Notice that we could write $z(t) = x(t) \cdot y(t)$. Find $Z(\omega)$ by using this insight and tables in Appendix 4.A and 4.B.

(b) Notice also that the signal $z(t)$ is absolutely integrable and its Fourier transform is well-defined. Find $Z(\omega)$ directly by using the Fourier transform equation.

(c) Confirm that part (a) is indeed the same as part (b).

Problem 2: Properties of Fourier transform

Appendix 4.A in the lecture notes summarizes the properties of continuous-time Fourier transform.

(a) Assume that $x_1(t) \circ\!\!-\!\!\bullet X_1(\omega)$ and $x_2(t) \circ\!\!-\!\!\bullet X_2(\omega)$. Express the Fourier transform of the signal

$$y(t) \quad = \quad x_1(t-3) + x_2\left(\frac{t}{2}\right)$$

in terms of $X_1(\omega)$ and $X_2(\omega)$ using Appendix 4.A in the lecture notes.

(b) Assume that $x(t) \circ\!\!-\!\!\bullet X(\omega)$, $y(t) \circ\!\!-\!\!\bullet Y(\omega)$ and $z(t) \circ\!\!-\!\!\bullet Z(\omega)$. For two absolutely integrable real-valued functions $x(t)$ and $y(t)$, express the continuous-time Fourier transform $Z(\omega)$ of the signal

$$z(t) \quad = \quad \int_{-\infty}^{\infty} x(\tau) y(t+\tau) d\tau$$

in terms of $X(\omega)$ and $Y(\omega)$.

(c) Assume that $x(t) \circ\!\!-\!\!\bullet X(\omega)$. Express the Fourier transform of the signal

$$y(t) \quad = \quad x(2t-5)$$

in terms of $X(\omega)$.

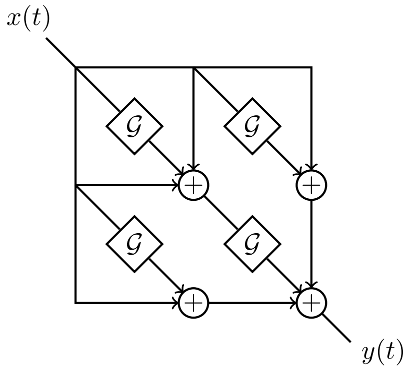
(d) Assume that $x_1(t) \circ\!\!-\!\!\bullet X_1(\omega)$ and $x_2(t) \circ\!\!-\!\!\bullet X_2(\omega)$. Find the inverse Fourier transform of the function

$$Y(\omega) \quad = \quad 2X_1^*(\omega - 5) + 2\frac{d}{d\omega} X_2(\frac{\omega}{2})$$

in terms of $x_1(t)$ and $x_2(t)$ using Appendix 4.A in the lecture notes.

Problem 3: Composition of Systems

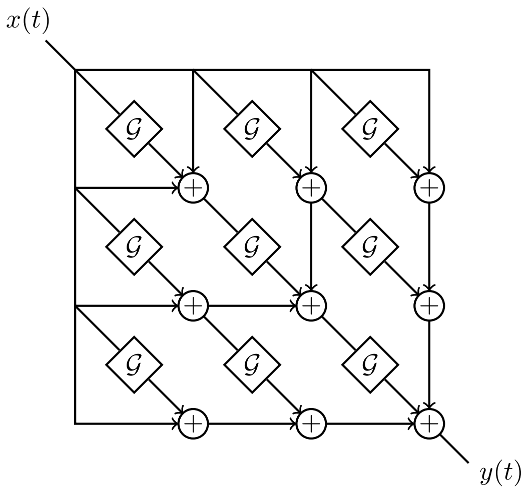
Assume that the system $\mathcal{G}$ is stable and linear time-invariant with impulse response $g(t)$. We combine many systems $\mathcal{G}$ in a systematic manner as follows.



Find the overall impulse response of the system with input $x(t)$ and output $y(t)$ in terms of $g(t)$.

(Optional Additional Challenge: The system in the figure can be interpreted as a 2×2 composition, and we can naturally extend it to a 3×3 composition, and further to a general $D \times D$ composition. Find the corresponding overall impulse response $h_D(t)$.

Hint: Start with $D = 3$ to observe the pattern as depicted in the figure below. If $h_D(t)$ is the overall impulse response of a $D \times D$ concatenation, then try to write $h_D(t)$ in terms of $h_{D-1}(t)$.)



Problem 4: Directly computing discrete-time Fourier Transform

Use the discrete-time Fourier transform equation to find the Fourier transform of the following signals:

(a) $x[n] = \frac{j}{2}\delta[n+1] - \frac{j}{2}\delta[n-1]$,

(b) $x[n] = -\alpha^n u[-n-1]$ where $|\alpha| > 1$.

Use the discrete-time inverse Fourier transform equation to find the inverse Fourier transform of the following signal:

(c) $X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \left(2\pi\delta(\omega - 2\pi k) + \pi\delta(\omega - \frac{\pi}{3} - 2\pi k) + \pi\delta(\omega + \frac{\pi}{3} - 2\pi k)\right)$.

Problem 5: LTI Systems : input/output relationship

One of the main results of this class is that (stable) LTI systems can either be tackled in the time domain or in the frequency (Fourier) domain. In the time domain, every LTI system can be characterized by its impulse response, and the input-output relationship can be written as $y(t) = (h * x)(t)$. In the frequency domain, every stable LTI system can be characterized by its frequency response, and the input-output relationship can be written as $Y(\omega) = H(\omega)X(\omega)$. Depending on the task and the signals at hand, one of the two representations can be substantially simpler than the other. As you become more proficient in the use of these fundamental tools, you will start to develop an intuition for which one to select, but in general, it definitely pays off to briefly consider both approaches, and then decide which one to select. In the following, feel free to make use of the formulas in Appendices 4.A and 4.B in the lecture notes. Find $y(t)$.

(a) $h(t) = e^{-at}u(t)$, $x(t) = e^{-bt}u(t)$, where $a > b > 0$.

(b) $h(t) = \frac{8}{\pi}\mathrm{sinc}(\frac{8}{\pi}(t-2))$, and $x(t) = \frac{1}{\pi}\left(\mathrm{sinc}(\frac{1}{\pi}t)\right)^2$.

Problem 6: Discrete-time Fourier transform properties

Given that $x[n]$ has Fourier transform $X(e^{j\omega})$, express the Fourier transform of the following signals in terms of $X(e^{j\omega})$:

(a) $y[n] = (n - n_0)x[n - n_0]$,

(b) $y[n] = x[n]\cos(\omega_0 n)$,

(c) $y[n] = (-1)^n x[n]$,

Problem 7: System characterization (discrete)

Suppose we are given one input/output pair of an unknown system. Namely, if the input signal is

$$x[n] = \left(\frac{1}{3}\right)^n u[n] - \frac{4}{3}\left(\frac{1}{3}\right)^{n-1} u[n-2],$$

then we observe the following output:

$$y[n] = \left(\frac{4}{5}\right)^n u[n].$$

(a) Find the impulse response and the frequency response of a discrete-time LTI system that gives this output to the mentioned input.

Hint: $H\left(e^{j\omega}\right) = \frac{A}{1-\frac{2}{3}e^{-j\omega}} + \frac{B}{1+\frac{2}{3}e^{-j\omega}} + \frac{C}{1-\frac{4}{5}e^{-j\omega}}$, where $A = -\frac{5}{4}$, $B = \frac{15}{44}$ and $C = \frac{21}{11}$. The hint is given to help you out with the calculation. It is your duty to prove also the hint.

(b) Find a difference equation that relates $x[n]$ and $y[n]$.