Solution Set 1

Solution 1: Basic signals operations

(a) Consider the complex-valued signal $y(t) = Ae^{j\omega_0 t}$ where $A \neq 0$ and ω_0 are real numbers. Express (as functions of t): $Re\{y(t)\}$, $Im\{y(t)\}$, $|y(t)|$, and $\arg(y(t))$.

(b) Consider the following function:

$$x[n] = \begin{cases} 1, & 0 \leq n \leq 2, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Sketch $y[n] = x[n-1] + x[2n] + x[-1-n]$. Carefully label both axes in the plot.

(c) Consider the following function:

$$x(t) = \begin{cases} 1, & -1 \leq t \leq 1, \\ 2-t, & 1 < t \leq 2, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Sketch $y(t) = x(t) + x(t/2 + 2)$. Carefully label both axes in the plot.

Solution

(a) We have $y(t) = A\cos(\omega_0 t) + Aj\sin(\omega_0 t)$. It is straightforward to see that

$$Re\{y(t)\} = A\cos(\omega_0 t), \quad Im\{y(t)\} = A\sin(\omega_0 t), \quad |y(t)| = |A|.$$

If $A > 0$, then

$$\arg(y(t)) = (\omega_0 t) \bmod 2\pi.$$

If $A < 0$, then $y(t) = Ae^{j\omega_0 t} = |A|e^{j(\omega_0 t + \pi)}$ and

$$\arg(y(t)) = (\omega_0 t + \pi) \bmod 2\pi.$$

(b) The plot is shown in Figure 1.

(c) The plot is shown in Figure 2.

Solution 2: Properties of signals

(a) Determine whether or not each of the following discrete-time signals is periodic. If the signal is periodic, determine its fundamental period.

$$x[n] = \cos\left(\frac{6\pi}{7}n + 1\right), \quad x[n] = \sin\left(\frac{\pi}{2}n\right)\cos\left(\frac{\pi}{4}n\right), \quad x[n] = \cos\left(\frac{n}{8} - \pi\right) \tag{3}$$

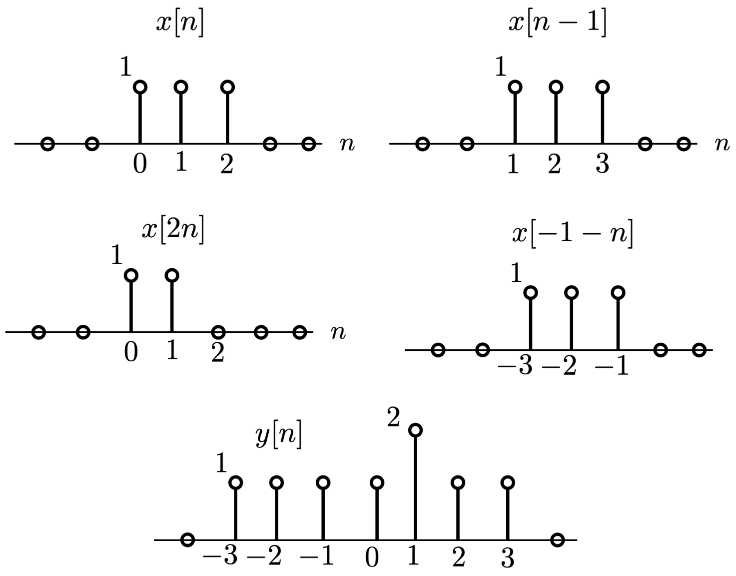


Figure 1: Problem 2 (b).

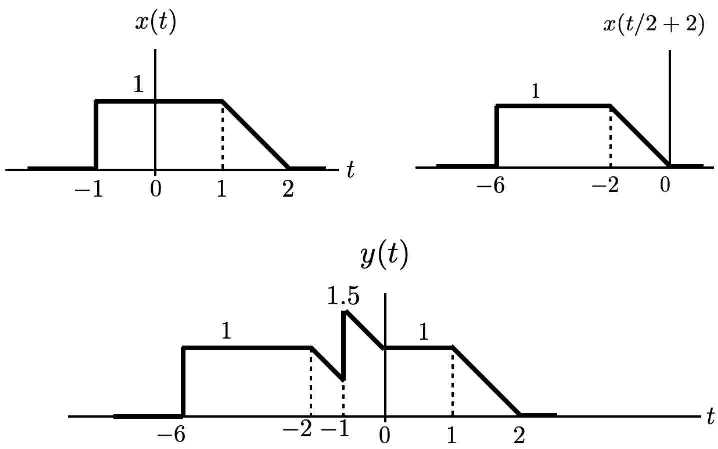


Figure 2: Problem 2 (c).

Categorize each of the following signals as an energy signal or a power signal, and find the energy or time-averaged power of the signal: (Please include your derivation of the result.)

(b) The discrete-time signal $y[n]$, defined by

$$y[n] \quad = \quad \begin{cases} n, & 0 \leq n \leq 6, \\ 2, & 6 < n \leq 8, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

(c) The continuous-time signal $z(t)$, defined for $-\infty < t < \infty$ by

$$z(t) \quad = \quad \sum_{k=-\infty}^{\infty} A_k g(t - kT), \tag{5}$$

where $A_k = \sqrt{2}$ if $|k|$ is a prime number and $A_k = -\sqrt{2}$ otherwise, and where $g(t)$ is the "square pulse function," *i.e.,* $g(t) = 1$ for $0 \leq t < T$ and $g(t) = 0$ otherwise. *Hint:* Sketch $z(t)$ for $0 \leq t \leq 4T$.

Solution

(a) The signal $x[n] = \cos\left(\frac{6\pi}{7}n + 1\right)$ is periodic with the fundamental period $N = 7$. This could be found by writing $\cos\left(\frac{6\pi}{7}n + 1\right) = \cos\left(\frac{6\pi}{7}(n+N) + 1\right)$. Noting that the cosine has a period of 2π we obtain $\frac{6\pi}{7}N = 2\pi m$ for some integer m. The smallest integer N that gives a solution to this equation is $N = 7$ (with $m = 3$).

The signal $x[n] = \sin\left(\frac{\pi}{2}n\right)\cos\left(\frac{\pi}{4}n\right)$ is periodic with the fundamental period $N = 8$. Note that $\sin\left(\frac{\pi}{2}n\right)$ is periodic with the fundamental period $N_1 = 4$ and $\cos\left(\frac{\pi}{4}n\right)$ is periodic with the fundamental period $N_2 = 8$. The overall fundamental period is the least common multiple of N_1 and N_2.

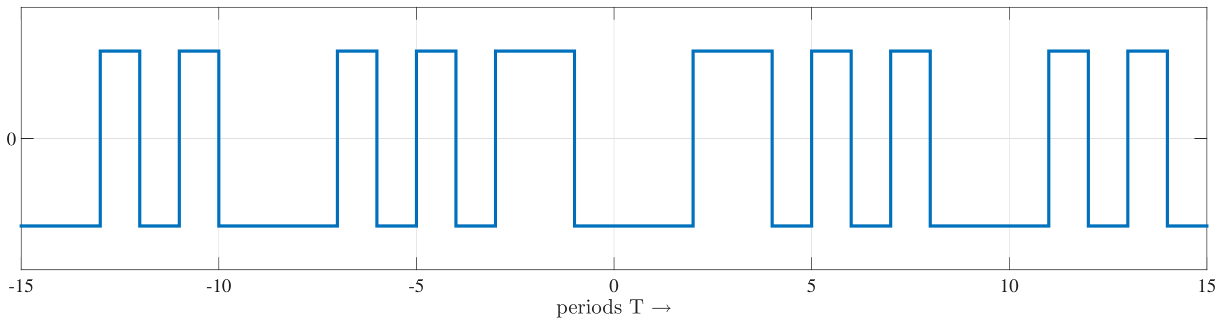
The signal $x[n] = \cos\left(\frac{n}{8} - \pi\right)$ is not periodic since the equation $\frac{N}{8} = 2\pi$ does not have integer solutions.

(b) The signal $y[n]$ is only non-zero for $n \in [0, 8]$ and takes on only finite values. Hence,

$$\mathcal{E} = \sum_{n=-\infty}^{+\infty} |y[n]|^2 = \sum_{n \in \{0,\dots,6\}} |n|^2 + \sum_{n \in \{7,8\}} |2|^2 = 0^2 + 1^2 + \cdots + 6^2 + 2 \cdot 2^2 = 99,$$

and since this value is finite, $y[n]$ is an *energy* signal.

(c) First of all, one should notice immediately that $z(t)$ does not go to zero as $t \to \infty$. Hence, if we take the integral of its absolute value it can never be finite and as of such $z(t)$ cannot be an energy signal; it must be a power signal. To be precise, $z(t)$ looks as follows:



The signal is a square wave with changing sign, but notice that the time-averaged power never changes. Namely, to compute the power we evaluate the absolute value $|z(t)|^2$, which is a straight line!

$$\begin{aligned}\mathcal{P} &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} 2dt \\ &= 2.\end{aligned}$$

Solution 3: Properties of systems

Are the following systems linear? Are they time-invariant? In each case, **give a full justification** using the definitions of these properties.

(a) $\mathcal{H}\{x(t)\} = x(t-b),$ where b can be any non-zero real number (positive or negative)

(b) $\mathcal{H}\{x(t)\} = x(t) - b,$ where b can be any non-zero real number (positive or negative)

(c) $\mathcal{H}\{x(t)\} = x(t^2 - 1)$,

(d) $\mathcal{H}\{x(t)\} = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$, where $h(t)$ is some arbitrary function.

Solution

Following the Lecture Notes, we use $y(t)$ to denote $\mathcal{H}(x(t))$.

(a) This system is linear, as for two signals $x_1(t)$ and $x_2(t)$ and two constants c_1, c_2, we have

$$\mathcal{H}\{c_1x_1(t) + c_2x_2(t)\} = c_1x_1(t-b) + c_2x_2(t-b) = c_1\mathcal{H}\{x_1(t)\} + c_2\mathcal{H}\{x_2(t)\}.$$

To tackle time-invariance is quite difficult, as this function might easily confuse you. Let us first try delaying the input by τ, to that end define

$$x'(t) = x(t-\tau).$$

Then we have

$$\mathcal{H}\{x'(t)\} = x'(t-b) = x(t-b-\tau).$$

On the other hand, delaying the output has the same output:

$$y(t-\tau) = \mathcal{H}\{x(t-\tau)\} = x(t-\tau-b)$$

Hence, the system is time-invariant.

(b) This system is *not* linear, as for two signals $x_1(t)$ and $x_2(t)$ and two constants c_1, c_2, we have

$$\mathcal{H}\{c_1x_1(t) + c_2x_2(t)\} = c_1x_1(t) + c_2x_2(t) - b.$$

In general, this does not equal to

$$c_1\mathcal{H}\{x_1(t)\} + c_2\mathcal{H}\{x_2(t)\} = c_1(x_1(t) - b) + c_2(x_2(t) - b) = c_1x_1(t) + c_2x_2(t) - (c_1 + c_2)b.$$

It is time-invariant. For any non-zero τ, we have

$$y(t-\tau) = x(t-\tau) - b,$$

while if we delay the input (and let us again use $x'(t) := x(t-\tau)$) we get

$$\mathcal{H}\{x'(t)\} = x'(t) - b = x(t-\tau) - b.$$

Evidently, $y(t-\tau) = \mathcal{H}\{x'(t)\}$.

(c) This system is linear. To see this, for two signals $x_1(t)$ and $x_2(t)$ and two constants c_1, c_2, we have

$$\mathcal{H}\{c_1x_1(t) + c_2x_2(t)\} = c_1x_1(t^2-1) + c_2x_2(t^2-1) = c_1\mathcal{H}\{x_1(t)\} + c_2\mathcal{H}\{x_2(t)\}.$$

This system is not time-invariant since if we delay the input (and let us again use $x'(t) := x(t-\tau)$) we get

$$\mathcal{H}\{x'(t)\} = x'(t^2-1) = x((t-\tau)^2 - 1) = x(t^2 - 2t\tau + \tau^2 - 1).$$

while

$$y(t-\tau) = x(t^2 - 1 - \tau)$$

(d) This system is linear, as for two signals $x_1(t)$ and $x_2(t)$ and two constants c_1, c_2, we have

$$\begin{aligned}\mathcal{H}\{c_1x_1(t) + c_2x_2(t)\} &= \int_{-\infty}^{\infty} (c_1x_1(t-\tau) + c_2x_2(t-\tau))\, h(\tau)d\tau \\ &= c_1 \int_{-\infty}^{\infty} x_1(t-\tau)h(\tau)d\tau + c_2 \int_{-\infty}^{\infty} x_2(t-\tau)h(\tau)d\tau \\ &= c_1\mathcal{H}\{x_1(t)\} + c_2\mathcal{H}\{x_2(t)\}.\end{aligned}$$

It is also time-invariant since for every number $\tilde{\tau}$, it holds that

$$y(t-\tilde{\tau}) = \int_{-\infty}^{\infty} x((t-\tilde{\tau}) - \tau)h(\tau)d\tau = \mathcal{H}\{x(t-\tilde{\tau})\}.$$

Solution 4: Properties of systems II

(a) Consider a discrete-time system with input $x[n]$ and output $y[n]$ related by

$$y[n] = \sum_{k=n-n_0}^{n+n_0} x[k]$$

where n_0 is a finite positive integer.

Determine whether the system is (i) linear, (ii) time-invariant, (iii) memoryless, (iv) causal, and (v) stable. In each case, give a short justification using the definitions of these properties.

A linear continuous-time system $\mathcal{H}\{\cdot\}$ yields the following input-output pairs:

$$e^{j3t} = \mathcal{H}\{e^{j2t}\} \quad \text{and} \quad e^{-j3t} = \mathcal{H}\{e^{-j2t}\}.$$

(b) If $x_1(t) = \cos(2t)$, determine the corresponding output $y_1(t) = \mathcal{H}\{x_1(t)\}$.

(c) If $x_2(t) = \cos(2(t - \frac{1}{2}))$, determine the corresponding output $y_2(t) = \mathcal{H}\{x_2(t)\}$.

Solution

(a) The system is linear, time-invariant, and stable. It is not memoryless or causal:

(i) linearity

Consider any signals $x_1[n]$, $x_2[n]$, and any real number a_1, a_2.

$$\begin{aligned}\mathcal{H}\{a_1x_1[n] + a_2x_2[n]\} &= \sum_{k=n-n_0}^{n+n_0} (a_1x_1[k] + a_2x_2[k]) \\ &= a_1 \sum_{k=n-n_0}^{n+n_0} x_1[k] + a_2 \sum_{k=n-n_0}^{n+n_0} x_2[k]) \\ &= a_1\mathcal{H}\{x_1[n]\} + a_2\mathcal{H}\{x_2[n]\}.\end{aligned}$$

(ii) time-invariance

Consider any integer m_0:

$$\mathcal{H}\{x[n-m_0]\} = \sum_{k=n-n_0}^{n+n_0} x[k-m_0] = \sum_{\tilde{k}=(n-m_0)-n_0}^{(n-m_0)+n_0} x[\tilde{k}] = y[n-m_0].$$

(v) stability

Assume that $|x[n]| < B < \infty$ for all n. Then,

$$|y[n]| = \sum_{k=n-n_0}^{n+n_0} x[k] < B(2n_0+1).$$

Since the output of the system depends on future time $n+n_0$, the system is not memoryless or causal.

(b) Using Euler's relation we write $x_1(t) = \frac{1}{2}e^{j2t} + \frac{1}{2}e^{-j2t}$. Then, using linearity we write

$$y_1(t) = \mathcal{H}\{x_1(t)\} = \mathcal{H}\left\{\frac{1}{2}e^{j2t} + \frac{1}{2}e^{-j2t}\right\} = \frac{1}{2}\mathcal{H}\left\{e^{j2t}\right\} + \frac{1}{2}\mathcal{H}\{e^{-j2t}\} = \frac{1}{2}e^{j3t} + \frac{1}{2}e^{-j3t} = \cos 3t.$$

(c) Using Euler's relation we write $x_2(t) = \frac{1}{2}e^{-j}e^{j2t} + \frac{1}{2}e^{j}e^{-j2t}$. Then, using linearity we write

$$y_2(t) = \mathcal{H}\{x_2(t)\} = \mathcal{H}\left\{\frac{1}{2}e^{-j}e^{j2t} + \frac{1}{2}e^{j}e^{-j2t}\right\} = \frac{1}{2}e^{-j}\mathcal{H}\left\{e^{j2t}\right\} + \frac{1}{2}e^{j}\mathcal{H}\{e^{-j2t}\} = \cos(3t-1).$$

Note that it is NOT correct to time-shift $y_1(t)$ to obtain $y_2(t)$ since we are not given that the system is time-invariant. In fact, from part (c) we see that it is actually not time variant.

Solution 5: Impulse response

Consider a discrete-time system from Problem 1 with input $x[n]$ and output $y[n]$ related by

$$y[n] = \sum_{k=n-n_0}^{n+n_0} x[k]$$

where n_0 is a finite positive integer. Find the impulse response $h[n]$ of this system.

Solution

The impulse response is given by

$$\begin{aligned} h[n] &= \sum_{k=n-n_0}^{n+n_0} \delta[k] \\ &= \left\{ \begin{array}{ll} 1, & -n_0 \le n \le n_0 \\ 0, & \text{otherwise.} \end{array} \right. \end{aligned}$$

Solution 6: Convolution

Calculate the convolution $(h*x)[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$, where

$$h[n] = \begin{cases} 1 & \text{if } n=0, \\ 2 & \text{if } n=3, \\ 0 & \text{otherwise,} \end{cases}$$

and $x[n] = u[n-1]$ where $u[n]$ is the unit step we defined in lecture.

Solution

$$\begin{aligned}(h*x)[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= h[0]x[n-0] + h[3]x[n-3] \\ &= u[n-1] + 2u[n-4] \\ &= \begin{cases} 1 & \text{if } n = \{1,2,3\} \\ 3 & \text{if } n \geq 4, \\ 0 & \text{otherwise,} \end{cases}\end{aligned}$$

Solution 7: Properties of systems, the differential operator

A system of some importance for the rest of this class is the differential operator: For a differentiable input signal $x(t)$, the output is given by $y(t) = \frac{d}{dt}x(t)$.

(a) Determine whether this system is linear (i) linear, and (ii) time-invariant.

(b) Determine whether this system is (i) memoryless, (ii) causal, and (iii) stable. *Hint:* Write out the definition of the derivative as you have learned it in your Analysis class:

$$\frac{d}{dt}x(t) = \lim_{m \to 0} \frac{x(t) - x(t-m)}{m}.$$

Solution

(a)

(i) This system is linear, as for two signals $x_1(t)$ and $x_2(t)$ and two constants c_1, c_2, we have

$$\begin{aligned}\mathcal{H}\{c_1x_1(t) + c_2x_2(t)\} &= \frac{d}{dt}(c_1x_1(t) + c_2x_2(t)) \\ &= c_1\frac{d}{dt}x_1(t) + c_2\frac{d}{dt}x_2(t) \\ &= c_1\mathcal{H}\{x_1(t)\} + c_2\mathcal{H}\{x_2(t)\}.\end{aligned}$$

(ii) It is also time-invariant since for every number τ, it holds that

$$y(t-\tau) = \frac{d}{dt}x(t-\tau) = \mathcal{H}\{x(t-\tau)\}.$$

(b)

(i)The system is not memoryless. As you have learned in linear algebra the derivative of a function does not only depend on the value of the function at time instant t but also in the ϵ-neighborhood of t. For instance, compare the two signals $x_1(t) = t$ and $x_2(t) = t^2$. They are equal at $t = 0$, therefore if derivative was a memoryless operator, we would expect them to also have equal derivatives at $t = 0$. However we know that derivative of x_1 at $t = 0$ is equal to 1 but derivative of x_2 at $t = 0$ is zero. Another way to look at it is by means of the definition of the derivative operator:

$$\frac{d}{dt}x(t) = \lim_{m \to 0} \frac{x(t) - x(t-m)}{m}.$$

Clearly, this expression does not only depend on t but also on the history of the signal.

(ii) The system is causal since the derivative operator is only defined for differentiable signals. For such signals the left and right derivatives are equal and thus the derivative is uniquely determined by the left derivative (which is clearly a casual operator).

(iii) This is true for most signals of practical interest, but unfortunately not in general. A simple counter example is the signal $x(t) = \sqrt{1-t^2}$ defined over $-1 \le t \le 1$. The input is bounded, however the derivative $y = \frac{-t}{\sqrt{1-t^2}}$ can get arbitrarily large as t approaches ± 1. In other words there is no B such that for $-1 < x < 1$ we have $|y| < B$. Therefore for this particular signal the system is not stable. One should keep in mind however that most functions that appear in nature (such as electric discharge of a capacitor or the position of a baseball hit by a bat) have smooth variations over time. Unlimited growth of the derivative only appears due to the simplified models that we use to describe those events.

Solution 8: Analytical convolution

Calculate analytically the convolution of the following pairs of signals:

(a) $h(t) = e^{-2t}u(t)$ and

$$x(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

(b) $x[n] = \alpha^n u[n]$ and $h[n] = \beta^n u[n]$, $\alpha, \beta \neq 0$

Solution

(a)

$$\begin{aligned}(x*h)(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{t-1}^{t} e^{-2\tau}u(\tau)d\tau\end{aligned}$$

The last step follows from the fact that we integrate over τ but $x(t-\tau) = 0$ for all $\tau < t-1$ and $\tau > t$, which renders the entire integral to be zero. Therefore we only need to integrate from $t-1$ to t. We still have to deal with $u(\tau)$, and can consider three separate cases:

1. If $t < 0$, then

$$(x*h)(t) = \int_{t-1}^{t} e^{-2\tau}u(\tau)d\tau = 0$$

since $u(\tau) = 0$ for τ in the interval $(t-1, t)$.

2. If $0 \le t < 1$, then

$$\begin{aligned}(x*h)(t) &= \int_{t-1}^{t} e^{-2\tau}u(\tau)d\tau \\ &= \int_{0}^{t} e^{-2\tau}d\tau \\ &= -\frac{1}{2}e^{-2\tau}\Big|_0^t = -\frac{1}{2}e^{-2t} + \frac{1}{2}.\end{aligned}$$

3. If $t \geq 1$, then

$$\begin{aligned}(x*h)(t) &= \int_{t-1}^{t} e^{-2\tau}u(\tau)d\tau \\ &= \int_{t-1}^{t} e^{-2\tau}d\tau \\ &= -\frac{1}{2}e^{-2\tau}\Big|_{t-1}^{t} = \frac{1}{2}e^{-2(t-1)} - \frac{1}{2}e^{-2t}.\end{aligned}$$

(b)

$$(x*h)[n] = \sum_{k=-\infty}^{\infty} \alpha^k u[k]\beta^{n-k}u[n-k]$$

Assuming $\alpha \neq \beta$,

$$\begin{aligned}(x*h)[n] &= \beta^n \sum_{k=0}^{\infty} \left(\frac{\alpha}{\beta}\right)^k u[n-k] \\ &= \beta^n \sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k \\ &= \beta^n \left(\frac{1-\left(\frac{\alpha}{\beta}\right)^{n+1}}{1-\left(\frac{\alpha}{\beta}\right)}\right) u[n].\end{aligned}$$

On the other hand, assuming $\alpha = \beta$,

$$\begin{aligned}(x*h)[n] &= \beta^n \sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k \\ &= \beta^n (n+1)u[n].\end{aligned}$$

Solution 9: Graphical convolution

Graphically convolve (flip-and-drag method) the following pairs of signals. You do not need to write the equations for your results, but clearly label and scale your axes. We focus our attention on two very particular waveforms: the step function and the rectangle/pulse. Let us define them as follows:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{6}$$

and

$$\text{rect}(t) = \begin{cases} 1 & 0 < t \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

For the following signals, sketch (by hand) the input $x(t)$, the filter $h(t)$, and the convolution $(x*h)(t)$.

(a) $h(t) = \text{rect}(t)$, $x(t) = u(t)$

(b) $h(t) = u(t)$, $x(t) = u(t)$

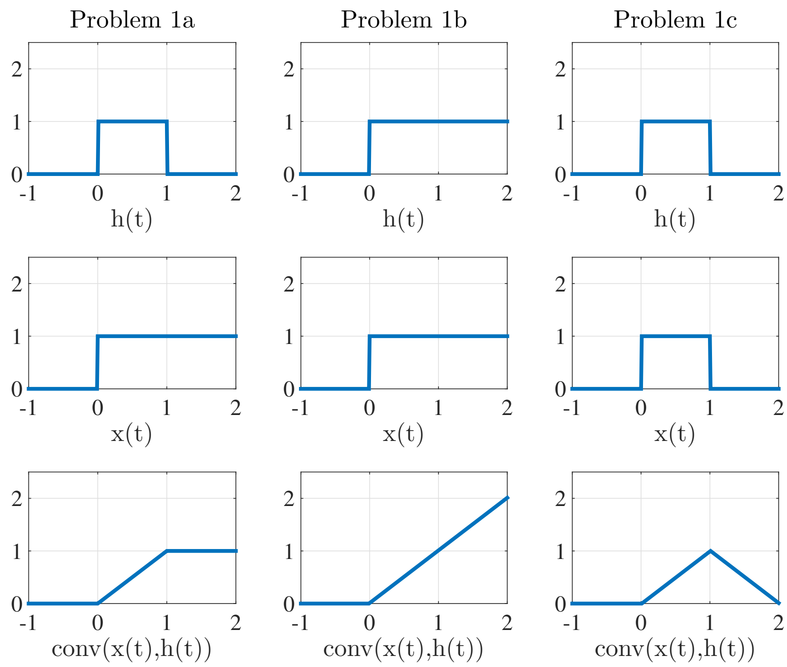
(c) $h(t) = \mathrm{rect}(t)\,,\ x(t) = \mathrm{rect}(t)$

Graphically convolve (flip-and-drag method) the following pair of signals. Again, drawings suffice, but make sure you properly scale and label the axes.

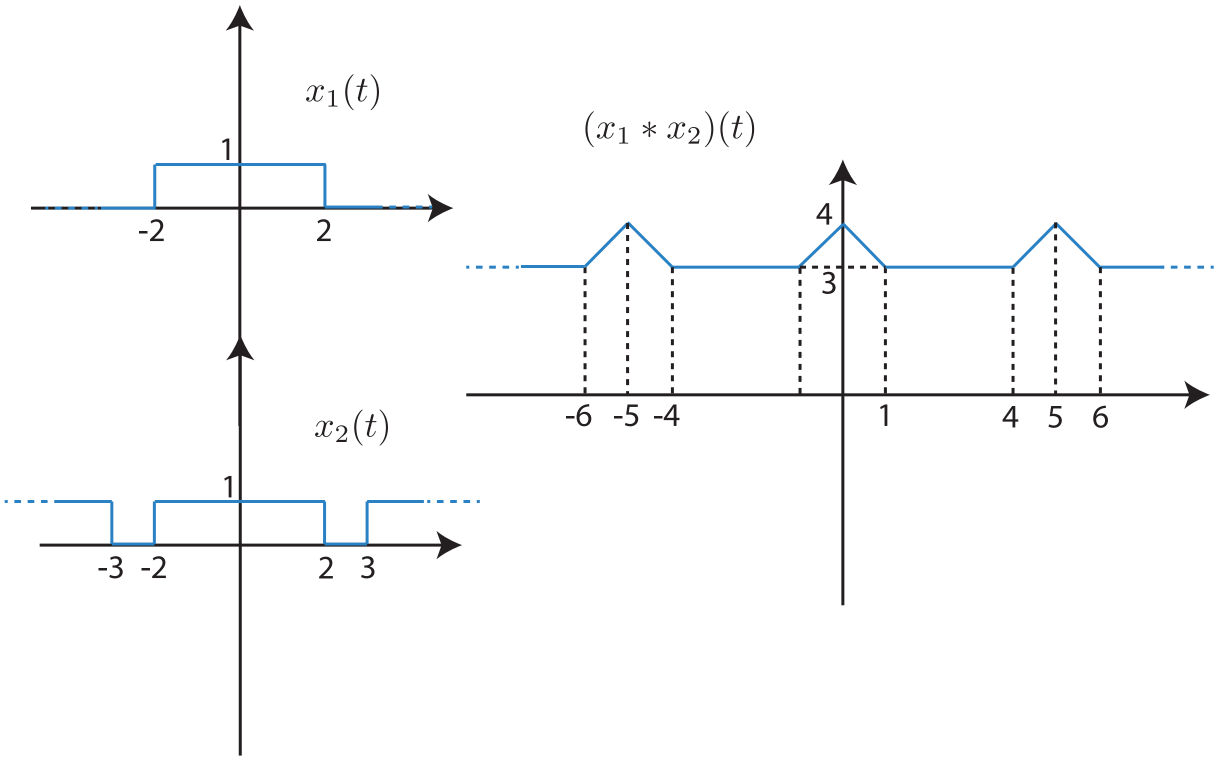
(d)

$$x_1(t) = \begin{cases} 1, & |t| \le 2 \\ 0, & |t| > 2 \end{cases} \qquad\qquad x_2(t) = \sum_{k=-\infty}^{\infty} x_1(t+5k)$$

Solution:



The convolved signal is periodic with the following plot showing only a part of it.



Solution 10: Composition of systems

Consider the following discrete-time LTI system $\mathcal{G}$, which is a composition of the discrete-time LTI systems $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$.

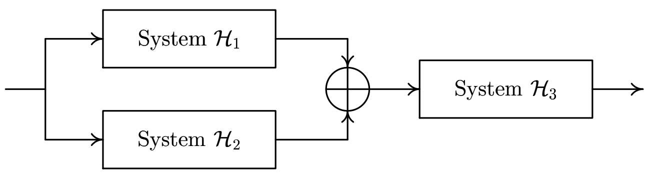


Figure 3: Composition of systems.

(a) Show that if $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$ are stable, then $\mathcal{G}$ is also stable.

Next, we consider causality. A discrete-time LTI system $\mathcal{H}$ is causal if and only if its impulse response $h[n]$ satisfies that $h[n] = 0$ for all $n < 0$.

(b) Assume that $\mathcal{H}_3\{x[n]\} = x[n]$. If the systems $\mathcal{H}_1$, $\mathcal{H}_2$ are non-causal, does it imply that $\mathcal{G}$ is also non-causal? If the statement is correct, give a proof; otherwise, provide a counterexample.

(c) Assume that $\mathcal{H}_2\{x[n]\} = 0$. If either $\mathcal{H}_1$ or $\mathcal{H}_3$ (not both) is non-causal, does it imply that $\mathcal{G}$ is non-causal? If the statement is correct, give a proof; otherwise, provide a counterexample.

Solution

(a) Consider a bounded signal $x[n]$, i.e., there exists $B \in [0, \infty)$ such that $|x[n]| \leq B$ for all $n \in \mathbb{Z}$. We show that the system $\mathcal{G}$ is stable in two steps:

1. Show that the parallel composition $\mathcal{F}\{x[n]\} = \mathcal{H}_1\{x[n]\} + \mathcal{H}_2\{x[n]\}$ is stable.
2. Show that the series composition $\mathcal{G}\{x[n]\} = \mathcal{H}_3\{\mathcal{F}\{x[n]\}\}$ is stable.

Step 1:
Since $\mathcal{H}_1$ and $\mathcal{H}_2$ are stable, there exist $B_1, B_2 \in [0, \infty)$ such that $|\mathcal{H}_1\{x[n]\}| < B_1$ and $|\mathcal{H}_2\{x[n]\}| < B_2$ for all $n \in \mathbb{Z}$, i.e., $\mathcal{H}_1\{x[n]\}$ and $\mathcal{H}_2\{x[n]\}$ are bounded signals. We have

$$\begin{aligned}|\mathcal{F}\{x[n]\}| &= |\mathcal{H}_1\{x[n]\} + \mathcal{H}_2\{x[n]\}| \\ &\overset{\dagger}{\leq} |\mathcal{H}_1\{x[n]\}| + |\mathcal{H}_2\{x[n]\}| \\ &< B_1 + B_2\end{aligned}$$

where ($\dagger$) follows from the triangle inequality.

Step 2:
Additionally, since $\mathcal{H}_3$ is stable and the input signal $\mathcal{F}\{x[n]\}$ is bounded, there exists $B_3 \in [0, \infty)$ such that $|\mathcal{H}_3\{\mathcal{F}\{x[n]\}\}| < B_3$ for all $n \in \mathbb{Z}$. Thus, the system is stable.

(b) The answer is no. A simple counterexample is the following:

$$\begin{aligned}\mathcal{H}_1\{x[n]\} &= x[n+1] + x[n], \\ \mathcal{H}_2\{x[n]\} &= -x[n+1],\end{aligned}$$

then

$$\begin{aligned}\mathcal{G}\{x[n]\} &= \mathcal{H}_3\{\mathcal{H}_1\{x[n]\} + \mathcal{H}_2\{x[n]\}\} \\ &= \mathcal{H}_1\{x[n]\} + \mathcal{H}_2\{x[n]\} \\ &= x[n]\end{aligned}$$

in which the non-causal parts cancel each other out.

(c) Again, the answer is no. A simple counterexample is the following:

$$\begin{aligned}\mathcal{H}_1\{x[n]\} &= x[n-1], \\ \mathcal{H}_3\{x[n]\} &= x[n+1],\end{aligned}$$

where $\mathcal{H}_3$ is non-causal. To show that $\mathcal{G}$ is causal, note that both $\mathcal{H}_1$ and $\mathcal{H}_3$ are LTI systems. Therefore, to discuss the causality of $\mathcal{G}$, it suffices to look at its impulse response $g[n]$. Note that the impulse responses of the systems $\mathcal{H}_1$ and $\mathcal{H}_3$ are $h_1[n] = \delta[n-1]$ and $h_3[n] = \delta[n+1]$, respectively. Then, we have

$$\begin{aligned}g[n] &= (h_3 * h_1)[n] \\ &= \sum_{k=-\infty}^{\infty} h_3[k]h_1[n-k] \\ &= \sum_{k=-\infty}^{\infty} \delta[k+1]h_1[n-k] \\ &= h_1[n-(-1)] \\ &= h_1[n+1] \\ &= \delta[(n+1)-1] \\ &= \delta[n],\end{aligned}$$

which indicates that the system $\mathcal{G}$ is causal.

Solution 11: LTI Systems and Dirac delta

(a) An input signal $x(t) = u(t+1) - u(t-1)$ is applied to an LTI system with an impulse response $h(t) = 2^t\left(\delta(t) + \delta(t-T)\right)$, where T is some constant.

Find the output $y(t)$ of the system.

(b) Consider two LTI systems characterized by input-output relationships

(i)

$$y(t) = \int_0^\infty e^{-5\tau} x(t-2-\tau) d\tau, \tag{7}$$

(ii) and

$$y(t) = \int_{-\infty}^{t} \left(x\left(\tau+2\right) + x\left(\tau-2\right)\right) d\tau. \tag{8}$$

Find the impulse response of each of these systems.

Solution

(a) First note that we can write

$$h(t) = 2^t\left(\delta(t) + \delta(t-T)\right) = \delta(t) + 2^T\delta(t-T). \tag{9}$$

Since this is an LTI System, the output could be obtained by convolution with the impulse response. The easiest way to solve the resulting convolution problem is to remember that a convolution of any function with a shifted and scaled delta function returns the original function but shifted and scaled. Therefore:

$$(x * h)(t) = u(t+1) - u(t-1) + 2^T\left(u(t+1-T) - u(t-1-T)\right). \tag{10}$$

(b) Since this is an LTI System, the output could be obtained by convolution with the impulse response. The solution is to simply plug in the delta function in place of $x(t)$ and evaluate the integranls.

(i)

$$\begin{aligned} h(t) &= \int_0^\infty e^{-5\tau}\delta(t-2-\tau) d\tau && (11)\\ &= \int_0^\infty e^{-5(t-2)}\delta(t-2-\tau) d\tau && (12)\\ &= e^{-5(t-2)}\int_0^\infty \delta(t-2-\tau) d\tau && (13)\\ &= \begin{cases} e^{-5(t-2)} & \text{if } t > 2 \\ 0 & \text{if } t < 2 \end{cases} && (14) \end{aligned}$$

(ii)

$$\begin{aligned} h(t) &= \int_{-\infty}^{t} \left(\delta\left(\tau+2\right) + \delta\left(\tau-2\right)\right) d\tau && (15)\\ &= \int_{-\infty}^{t} \delta\left(\tau+2\right) d\tau + \int_{-\infty}^{t} \delta\left(\tau-2\right) d\tau && (16)\\ &= \begin{cases} 2 & \text{if } t > 2 \\ 1 & \text{if } -2 < t < 2 \\ 0 & \text{if } t < -2 \end{cases} && (17) \end{aligned}$$

Solution 12: Condition of initial rest

(a) Consider the first-order difference equation

$$y[n] - \frac{3}{2}y[n-1] = x[n].$$

Assuming the condition of initial rest (i.e. if $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$), find the impulse response $h[n]$ of an LTI system whose input and output are related by this difference equation.

(b) Is the system in part (a) stable? Why or why not?

(c) Consider the first-order difference equation

$$y[n] + \frac{1}{2}y[n-1] = x[n] + x[n-2].$$

Assuming that the system is causal (hint: this is the same as assuming the condition of initial rest), find the impulse response $h[n]$ of an LTI system whose input and output are related by this difference equation.

(d) Is the system in part (c) stable? Why or why not?

Solution

(a) We can rearrange the difference equation

$$y[n] = x[n] + \frac{3}{2}y[n-1].$$

The impulse response is given by

$$h[n] = \delta[n] + \frac{3}{2}h[n-1].$$

Using the condition of initial rest we see that

$$h[n] = 0$$

when $n < 0$ since $\delta[n] = 0$ for all $n < 0$. Furthermore,

$$\begin{aligned}
h[0] &= \delta[0] + \frac{3}{2}h[-1] = 1, \\
h[1] &= \delta[1] + \frac{3}{2}h[0] = \frac{3}{2}, \\
h[2] &= \delta[2] + \frac{3}{2}h[1] = \left(\frac{3}{2}\right)^2, \\
h[3] &= \delta[3] + \frac{3}{2}h[2] = \left(\frac{3}{2}\right)^3, \\
&\dots
\end{aligned}$$

The impulse response is

$$h[n] = \begin{cases} \left(\frac{3}{2}\right)^n, & n \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

(b) The system is not stable since $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n = \infty$.

(c) We can rearrange the difference equation

$$y[n] = x[n] + x[n-2] - \frac{1}{2}y[n-1].$$

The impulse response is given by

$$h[n] = \delta[n] + \delta[n-2] - \frac{1}{2}h[n-1].$$

Using the condition of initial rest we see that

$$h[n] = 0$$

when $n < 0$ since $\delta[n] = 0$ for all $n < 0$. Furthermore,

$$\begin{aligned}
h[0] &= \delta[0] + \delta[-2] - \frac{1}{2}h[-1] = 1, \\
h[1] &= \delta[1] + \delta[-1] - \frac{1}{2}h[0] = -\frac{1}{2}, \\
h[2] &= \delta[2] + \delta[0] - \frac{1}{2}h[1] = 1 + \frac{1}{4}, \\
h[3] &= \delta[3] + \delta[1] - \frac{1}{2}h[2] = -\frac{1}{2} - \frac{1}{8}, \\
&\cdots
\end{aligned}$$

The impulse response is $h[n] = \left(-\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{2}\right)^{n-2} u[n-2]$.

(d) The system is stable since $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^{n-2} = 4 < \infty$.