Problem Set 1

Problem 1: Basic signals operations

(a) Consider the complex-valued signal $y(t) = Ae^{j\omega_0 t}$ where $A \neq 0$ and ω_0 are real numbers.

Express (as functions of t): $Re\{y(t)\}$, $Im\{y(t)\}$, $|y(t)|$, and $\arg(y(t))$.

(b) Consider the following function:

$$x[n] \quad = \quad \begin{cases} 1, & 0 \leq n \leq 2, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Sketch $y[n] = x[n-1] + x[2n] + x[-1-n]$. Carefully label both axes in the plot.

(c) Consider the following function:

$$x(t) \quad = \quad \begin{cases} 1, & -1 \leq t \leq 1, \\ 2-t, & 1 < t \leq 2, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Sketch $y(t) = x(t) + x(t/2 + 2)$. Carefully label both axes in the plot.

Problem 2: Properties of signals

(a) Determine whether or not each of the following discrete-time signals is periodic. If the signal is periodic, determine its fundamental period.

$$x[n] = \cos\left(\frac{6\pi}{7}n + 1\right), \quad x[n] = \sin\left(\frac{\pi}{2}n\right)\cos\left(\frac{\pi}{4}n\right), \quad x[n] = \cos\left(\frac{n}{8} - \pi\right) \tag{3}$$

Categorize each of the following signals as an energy signal or a power signal, and find the energy or time-averaged power of the signal: (Please include your derivation of the result.)

(b) The discrete-time signal $y[n]$, defined by

$$y[n] \quad = \quad \begin{cases} n, & 0 \leq n \leq 6, \\ 2, & 6 < n \leq 8, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

(c) The continuous-time signal $z(t)$, defined for $-\infty < t < \infty$ by

$$z(t) \quad = \quad \sum_{k=-\infty}^{\infty} A_k g(t - kT), \tag{5}$$

where $A_k = \sqrt{2}$ if $|k|$ is a prime number and $A_k = -\sqrt{2}$ otherwise, and where $g(t)$ is the "square pulse function," *i.e.,* $g(t) = 1$ for $0 \leq t < T$ and $g(t) = 0$ otherwise. *Hint:* Sketch $z(t)$ for $0 \leq t \leq 4T$.

Problem 3: Properties of systems

Are the following systems linear? Are they time-invariant? In each case, **give a full justification** using the definitions of these properties.

(a) $\mathcal{H}\{x(t)\} = x(t-b)$, where b can be any non-zero real number (positive or negative)

(b) $\mathcal{H}\{x(t)\} = x(t) - b$, where b can be any non-zero real number (positive or negative)

(c) $\mathcal{H}\{x(t)\} = x(t^2 - 1)$,

(d) $\mathcal{H}\{x(t)\} = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$, where $h(t)$ is some arbitrary function.

Problem 4: Properties of systems II

(a) Consider a discrete-time system with input $x[n]$ and output $y[n]$ related by

$$y[n] = \sum_{k=n-n_0}^{n+n_0} x[k]$$

where n_0 is a finite positive integer.

Determine whether the system is (i) linear, (ii) time-invariant, (iii) memoryless, (iv) causal, and (v) stable. In each case, give a short justification using the definitions of these properties.

A linear continuous-time system $\mathcal{H}\{\cdot\}$ yields the following input-output pairs:

$$e^{j3t} = \mathcal{H}\{e^{j2t}\} \quad \text{and} \quad e^{-j3t} = \mathcal{H}\{e^{-j2t}\}.$$

(b) If $x_1(t) = \cos(2t)$, determine the corresponding output $y_1(t) = \mathcal{H}\{x_1(t)\}$.

(c) If $x_2(t) = \cos(2(t - \frac{1}{2}))$, determine the corresponding output $y_2(t) = \mathcal{H}\{x_2(t)\}$.

Problem 5: Impulse response

Consider a discrete-time system from Problem 1 with input $x[n]$ and output $y[n]$ related by

$$y[n] = \sum_{k=n-n_0}^{n+n_0} x[k]$$

where n_0 is a finite positive integer. Find the impulse response $h[n]$ of this system.

Problem 6: Convolution

Calculate the convolution $(h * x)[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$, where

$$h[n] \quad = \quad \begin{cases} 1 & \text{if } n = 0, \\ 2 & \text{if } n = 3, \\ 0 & \text{otherwise,} \end{cases}$$

and $x[n] = u[n-1]$ where $u[n]$ is the unit step we defined in lecture.

Problem 7: Properties of systems, the differential operator

A system of some importance for the rest of this class is the differential operator: For a differentiable input signal $x(t)$, the output is given by $y(t) = \frac{d}{dt}x(t)$.

(a) Determine whether this system is linear (i) linear, and (ii) time-invariant.

(b) Determine whether this system is (i) memoryless, (ii) causal, and (iii) stable. *Hint:* Write out the definition of the derivative as you have learned it in your Analysis class:

$$\frac{d}{dt}x(t) = \lim_{m \to 0} \frac{x(t) - x(t-m)}{m}.$$

Problem 8: Analytical convolution

Calculate analytically the convolution of the following pairs of signals:

(a) $h(t) = e^{-2t}u(t)$ and

$$x(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

(b) $x[n] = \alpha^n u[n]$ and $h[n] = \beta^n u[n]$, $\alpha, \beta \neq 0$

Problem 9: Graphical convolution

Graphically convolve (flip-and-drag method) the following pairs of signals. You do not need to write the equations for your results, but clearly label and scale your axes. We focus our attention on two very particular waveforms: the step function and the rectangle/pulse. Let us define them as follows:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{6}$$

and

$$\text{rect}(t) = \begin{cases} 1 & 0 < t \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

For the following signals, sketch (by hand) the input $x(t)$, the filter $h(t)$, and the convolution $(x * h)(t)$.

(a) $h(t) = \text{rect}(t)$, $x(t) = u(t)$

(b) $h(t) = u(t)$, $x(t) = u(t)$

(c) $h(t) = \text{rect}(t)$, $x(t) = \text{rect}(t)$

Graphically convolve (flip-and-drag method) the following pair of signals. Again, drawings suffice, but make sure you properly scale and label the axes.

(d)

$$x_1(t) = \begin{cases} 1, & |t| \leq 2 \\ 0, & |t| > 2 \end{cases} \qquad x_2(t) = \sum_{k=-\infty}^{\infty} x_1(t + 5k)$$

Problem 10: Composition of systems

Consider the following discrete-time LTI system $\mathcal{G}$, which is a composition of the discrete-time LTI systems $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$.

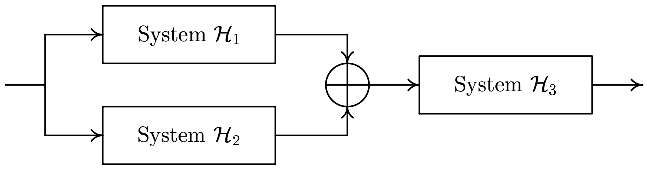

Figure 1: Composition of systems.

(a) Show that if $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$ are stable, then $\mathcal{G}$ is also stable.

Next, we consider causality. A discrete-time LTI system $\mathcal{H}$ is causal if and only if its impulse response $h[n]$ satisfies that $h[n] = 0$ for all $n < 0$.

(b) Assume that $\mathcal{H}_3\{x[n]\} = x[n]$. If the systems $\mathcal{H}_1$, $\mathcal{H}_2$ are non-causal, does it imply that $\mathcal{G}$ is also non-causal? If the statement is correct, give a proof; otherwise, provide a counterexample.

(c) Assume that $\mathcal{H}_2\{x[n]\} = 0$. If either $\mathcal{H}_1$ or $\mathcal{H}_3$ (not both) is non-causal, does it imply that $\mathcal{G}$ is non-causal? If the statement is correct, give a proof; otherwise, provide a counterexample.

Problem 11: LTI Systems and Dirac delta

(a) An input signal $x(t) = u(t+1) - u(t-1)$ is applied to an LTI system with an impulse response $h(t) = 2^t\left(\delta(t) + \delta(t-T)\right)$, where T is some constant.

Find the output $y(t)$ of the system.

(b) Consider two LTI systems characterized by input-output relationships

(i)

$$y(t) = \int_0^{\infty} e^{-5\tau} x(t-2-\tau)d\tau, \tag{7}$$

(ii) and

$$y(t) = \int_{-\infty}^{t} \left(x\left(\tau+2\right) + x\left(\tau-2\right)\right)d\tau. \tag{8}$$

Find the impulse response of each of these systems.

Problem 12: Condition of initial rest

(a) Consider the first-order difference equation

$$y[n] - \frac{3}{2}y[n-1] = x[n].$$

Assuming the condition of initial rest (i.e. if $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$), find the impulse response $h[n]$ of an LTI system whose input and output are related by this difference equation.

(b) Is the system in part (a) stable? Why or why not?

(c) Consider the first-order difference equation

$$y[n] + \frac{1}{2}y[n-1] = x[n] + x[n-2].$$

Assuming that the system is causal (hint: this is the same as assuming the condition of initial rest), find the impulse response $h[n]$ of an LTI system whose input and output are related by this difference equation.

(d) Is the system in part (c) stable? Why or why not?