Differential Geometry II - Smooth Manifolds
Winter Term 2024/2025
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Exercise Sheet 3

Exercise 1 (*Equivalent characterizations of smoothness*)**:**
Let M and N be smooth manifolds and let $F\colon M \to N$ be a map. Show that F is smooth if and only if either of the following conditions is satisfied:

(a) For every $p \in M$ there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $U \cap F^{-1}(V)$ is open in M and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi\big(U \cap F^{-1}(V)\big)$ to $\psi(V)$.

(b) F is continuous and there exist smooth atlases $\big\{(U_\alpha, \varphi_\alpha)\big\}$ and $\big\{(V_\beta, \psi_\beta)\big\}$ for M and N, respectively, such that for each α and β, $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is a smooth map from $\varphi_\alpha\big(U_\alpha \cap F^{-1}(V_\beta)\big)$ to $\psi_\beta(V_\beta)$.

Exercise 2 (*Smoothness is a local property*)**:**
Let M and N be smooth manifolds and let $F\colon M \to N$ be a map. Prove the following assertions:

(a) If every point $p \in M$ has a neighborhood U such that $F|_U$ is smooth, then F is smooth.

(b) If F is smooth, then its restriction to every open subset of M is smooth.

Exercise 3: Let M, N and P be smooth manifolds. Prove the following assertions:

(a) If $F\colon M \to N$ is a smooth map, then the coordinate representation of F with respect to every pair of smooth charts for M and N is smooth.

(b) If $c\colon M \to N$ is a constant map, then c is smooth.

(c) The identity map $\mathrm{Id}_M\colon M \to M$ is smooth.

(d) If $U \subseteq M$ is an open submanifold, then the inclusion map $\iota\colon U \hookrightarrow M$ is smooth.

(e) If $F\colon M \to N$ and $G\colon N \to P$ are smooth maps, then the composite $G \circ F\colon M \to P$ is also smooth.

Exercise 4:
Let $M_1, \dots, M_k$ be smooth manifolds. For each $i \in \{1, \dots, k\}$, let

$$\pi_i \colon \prod_{j=1}^{k} M_j \to M_i$$

be the projection onto the i-th factor.

(a) Show that each π_i is smooth.

(b) Let N be a smooth manifold. Show that a map $F \colon N \to \prod_{j=1}^{k} M_j$ is smooth if and only if each of the component maps $F_i := \pi_i \circ F \colon N \to M_i$ is smooth.

Exercise 5 (to be submitted by Thursday, 03.10.2024, 16:00):
Prove the following assertions:

(a) The quotient map $\pi \colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is smooth.

(b) A map $F \colon \mathbb{RP}^n \to M$ to a smooth manifold M is smooth if and only if the composite map $F \circ \pi \colon \mathbb{R}^{n+1} \setminus \{0\} \to M$ is smooth.

Exercise 6:
Show that the map

$$F \colon \mathbb{R}^n \to \mathbb{RP}^n, \ (x^1, \dots, x^n) \mapsto [1 : x^1 : \dots : x^n]$$

is a diffeomorphism onto a dense open subset of $\mathbb{RP}^n$.