Differential Geometry II - Smooth Manifolds
Winter Term 2024/2025
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Exercise Sheet 1 – Solutions

Exercise 1: Show that if a topological space M is locally Euclidean at some point $p \in M$ (i.e., p has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$), then p has a neighborhood that is homeomorphic to the whole space $\mathbb{R}^n$ or to an open ball in $\mathbb{R}^n$.

Solution: We know that there is an open neighborhood U of p and a homeomorphism φ from U to an open subset $\varphi(U)$ of $\mathbb{R}^n$. We can find a ball $B\big(\varphi(p), r\big) \subseteq \varphi(U) \subseteq \mathbb{R}^n$ for some $r > 0$. Consider now the map $\psi\colon B\big(\varphi(p), r\big) \to \mathbb{R}^n$ given by

$$\psi(x) := \frac{x - \varphi(p)}{r - \|x - \varphi(p)\|}.$$

One can easily verify that ψ is a homeomorphism with inverse

$$\psi^{-1}(y) = \varphi(p) + \frac{y}{1 + \|y\|}.$$

Set $U' := \varphi^{-1}\left(B\big(\varphi(p), r\big)\right) \subseteq M$ and observe that U' is a neighborhood of p in M. Then the map

$$\theta := \psi \circ \varphi|_{U'}\colon U' \to \mathbb{R}^n$$

is a homeomorphism, as both ψ and φ are homeomorphisms.

Exercise 2: Examine which of the following spaces (endowed with the subspace topology) is locally Euclidean:

(a) The closed interval $[0, 1] \subseteq \mathbb{R}$.

(b) The “bent line” $\big\{(x, y) \in \mathbb{R}^2 \mid x \geq 0,\ y \geq 0,\ xy = 0\big\} \subseteq \mathbb{R}^2$.

Solution:

(a) The interval $[0, 1]$ is *not* locally Euclidean. Suppose by contradiction that it is locally Euclidean. By *Exercise* 1 there is a neighborhood $U \subseteq [0, 1]$ of 0 which is homeomorphic to $\mathbb{R}^n$ for some $n \geq 1$. Denote by $\varphi\colon U \to \mathbb{R}^n$ a homeomorphism and note that U is connected, and thus of the form $U = [0, \varepsilon)$ for some $\varepsilon > 0$. But then $U \setminus \{0\} = (0, \varepsilon)$

is homeomorphic to $\mathbb{R}^n \setminus \{\varphi(0)\}$, and since $(0, \varepsilon)$ is still connected, we infer that $n > 1$ ($\mathbb{R}$ minus a point has two connected components). Now there are two ways to conclude: First, note that $(0, \varepsilon)$ and $\mathbb{R}^n \setminus \{\varphi(0)\}$ are topological manifolds of dimension 1 and n, respectively, and since the dimension of a topological manifold is a topological invariant, we obtain $n = 1$, a contradiction. Second, if $x \in (0, \varepsilon)$, then $(0, \varepsilon) \setminus \{x\}$ is homeomorphic to $\mathbb{R}^n \setminus \{\varphi(0), \varphi(x)\}$; as $n > 1$, the latter is connected, while the former is not, a contradiction.

(b) The "bent line"

$$L \coloneqq \big\{(x, y) \in \mathbb{R}^2 \mid x \geq 0,\ y \geq 0,\ xy = 0\big\}$$

is locally Euclidean. Indeed, denote by $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ the counterclockwise rotation around the origin by 45°. As this is a homeomorphism, we obtain that $L \cong \varphi(L)$. But now note that $\varphi(L)$ coincides with the graph of the absolute value function $|\bullet|\colon \mathbb{R} \to \mathbb{R}$. Thus, we obtain $L \cong \varphi(L) \cong \mathbb{R}$.

Exercise 3:

(a) *The line with two origins*: Consider the set

$$X = \big\{(x, y) \in \mathbb{R}^2 \mid y \in \{-1, 1\}\big\} \subseteq \mathbb{R}^2$$

and let M be the quotient of X by the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Show that M is locally Euclidean and second-countable, but not Hausdorff.

(b) Show that a disjoint union of uncountably many copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second-countable.

Solution:

(a) Denote by $\pi\colon X \to M$ the quotient map $(x, y) \mapsto [(x, y)]$. The two "origins" are the equivalence classes of the points $(0, y) \in X$ for $y = \pm 1$; these classes have just one element each and we denote them by $0_y = [(0, y)] = \{(0, y)\} \in M$. In contrast, the equivalence class of any other point $(x, y) \in X$ with $x \neq 0$ is the two-point set $\widetilde{x} = [(x, y)] = \{(x, 1), (x, -1)\} \in M$. Therefore, M is the set of equivalence classes

$$M = X/\sim = \{0_1\} \cup \{0_{-1}\} \cup \{\widetilde{x}\}_{x \neq 0}.$$

The space M is locally Euclidean of dimension 1 because it is the union of two open sets

$$\mathbb{R}_y = \big\{[(x, y)] \in M \mid x \in \mathbb{R}\big\} \quad (\text{for } y = \pm 1),$$

each of which is homeomorphic to $\mathbb{R}$ via the map

$$\begin{aligned} \varphi_y\colon \mathbb{R} &\to \mathbb{R}_y \\ x &\mapsto [(x, y)]. \end{aligned}$$

To see that the sets $\mathbb{R}_y$ are open in the quotient topology, note that

$$\pi^{-1}(\mathbb{R}_y) = X \setminus \big\{(0, -y)\big\},$$

which is open in X.

Moreover, M is second-countable because it is the union of two second-countable open subsets, namely, the sets $\mathbb{R}_y \cong \mathbb{R}$ (for $y = \pm 1$).

Finally, M is not Hausdorff: let U_{-1} be any open set containing 0_{-1} and let U_1 be any open set containing 0_1. For $y \in \{-1, 1\}$, as $\pi^{-1}(U_y)$ is an open subset of X containing $(0, y)$, it contains a set of the form $V_y = (-\varepsilon_y, \varepsilon_y) \times \{y\}$ for some $\varepsilon_y > 0$. Now let x be a real number such that $0 < x < \min\{\varepsilon_{-1}, \varepsilon_1\}$. Then $[(x, -1)] = [(x, 1)]$ is contained in both U_{-1} and U_1. Hence, 0_{-1} and 0_1 cannot be separated by disjoint open neighborhoods.

(b) Let I be an uncountable index set. For every $i \in I$ denote by $\mathbb{R}_i$ a copy of the real numbers $\mathbb{R}$ equipped with the Euclidean topology, and let

$$X := \bigsqcup_{i \in I} \mathbb{R}_i$$

be their disjoint union. Recall that there is a natural topology on X, defined as follows: For every i, denote by $f_i \colon \mathbb{R}_i \to X$ the natural set-theoretic inclusion. Then

$$\tau := \left\{ U \subseteq X \mid \forall i \in I : \, f_i^{-1}(U) \text{ open in } \mathbb{R}_i \right\}$$

is a topology on X; in fact, it is the finest (i.e. maximal) topology on X such that all the maps f_i are continuous.

To see that (X, τ) is Hausdorff, let $x, y \in X$ be arbitrary. Let $i, j \in I$ be such that $x \in f_i(\mathbb{R}_i)$ and $y \in f_j(\mathbb{R}_j)$. If $i \neq j$, then $f_i(\mathbb{R}_i)$ and $f_j(\mathbb{R}_j)$ are disjoint open neighborhoods of x and y, respectively (check this!). If $i = j$, then since $\mathbb{R}_i$ is Hausdorff, we can find disjoint open neighborhoods $U, V \subseteq \mathbb{R}_i$ separating (the preimages of) x and y in $\mathbb{R}_i$. Then $f_i(U)$ and $f_i(V)$ are disjoint open neighborhoods of x and y, respectively, inside X (again, check this!). As $x, y \in X$ were arbitrary, we conclude that X is Hausdorff.

Next, to check that X is locally Euclidean, let $x \in X$ be arbitrary. Let $i \in I$ be such that $x \in f_i(\mathbb{R}_i)$. Then $f_i(\mathbb{R}_i) \cong \mathbb{R}$ is a Euclidean open neighborhood of x inside X.

Finally, suppose by contradiction that X is second-countable, i.e. there exists a countable basis $\mathfrak{B}$ for its topology τ. Note that, for every $i \in I$, the set $f_i(\mathbb{R}_i)$ is open in X, and thus there exists $\emptyset \neq U_i \in \mathfrak{B}$ such that $U_i \subseteq f_i(\mathbb{R}_i)$. But then we must have $U_i \neq U_j$ for all $i \neq j$, and thus the map

$$I \to \mathfrak{B}, \; i \mapsto U_i$$

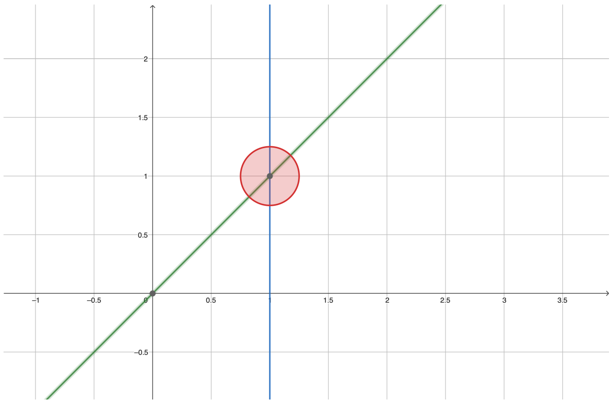
is an injection. However, since I is uncountable, this contradicts our hypothesis that $\mathfrak{B}$ is countable.

Exercise 4: Consider the subset

$$V = \left\{ (x, y) \in \mathbb{R}^2 \mid (x - 1)(x - y) = 0 \right\} \subseteq \mathbb{R}^2$$

endowed with the subspace topology. Show that V is not a topological manifold.

Solution: The subset $V \subseteq \mathbb{R}^2$ and a disc with small radius and centered at the point $(1, 1) \in \mathbb{R}^2$ (which is the point of intersection of the lines $y = x$ and $x = 1$) have been plotted below.



Since V is a subspace of $\mathbb{R}^2$, it is Hausdorff and second-countable. By considering any point $p \in V \setminus \{(1,1)\}$, we conclude that if V were a topological manifold, then it would necessarily have dimension 1. Assume now by contradiction that V is a topological 1-manifold. Then there exists an open neighborhood W of $(1,1)$ which is homeomorphic to an open subset G of $\mathbb{R}$; denote by φ this homeomorphism. For sufficiently small $\varepsilon > 0$, the set $U \coloneqq B\big((1,1),\varepsilon\big) \cap W$ (the red disc above) is an open neighborhood of $(1,1)$ in W, which is connected. Hence, its homeomorphic image $I \coloneqq \varphi(U)$ in $G \subseteq \mathbb{R}$ is connected as well, and thus $I \subseteq \mathbb{R}$ is an open interval. Observe now that $U \setminus \{(1,1)\}$ has four connected components, whereas $I \setminus \{\varphi(1,1)\}$ has only two connected components, a contradiction. In conclusion, V is not a topological manifold.

Exercise 5 (*Product manifolds*)**:** Let $M_1, \ldots, M_k$ be topological manifolds of dimensions $n_1, \ldots, n_k$, respectively, where $k \geq 2$. Show that the product space $M_1 \times \ldots \times M_k$ is a topological manifold of dimension $n_1 + \ldots + n_k$.

Solution: Any finite product of Hausdorff spaces is also Hausdorff: two distinct points of the product differ at some coordinate, where we can separate them by two disjoint neighborhoods. Moreover, if for each $1 \leq i \leq k$ we denote by $\mathcal{B}_i$ a countable basis for the topology of M_i, then

$$\mathcal{B} \coloneqq \big\{B_1 \times \cdots \times B_k \mid \forall 1 \leq i \leq k :\ B_i \in \mathcal{B}_i\big\}$$

is a countable basis for the topology of the product $M_1 \times \cdots \times M_k$. Finally, given any point $P = (p_1, \ldots, p_k) \in M_1 \times \cdots \times M_k$, by *Exercise 1* we know that for every $1 \leq i \leq k$ there exists an open neighborhood $U_i \subseteq M_i$ of p_i such that $U_i \cong \mathbb{R}^{n_i}$. Therefore, $U \coloneqq U_1 \times \cdots \times U_k$ is an open neighborhood of P such that $U \cong \mathbb{R}^{n_1 + \ldots + n_k}$. In conclusion, $M_1 \times \cdots \times M_k$ is a topological manifold of dimension $n_1 + \ldots + n_k$.