Differential Geometry II - Smooth Manifolds
Winter Term 2024/2025
Lecturer: Dr. N. Tsakanikas
Assistant: L. E. Rösler

Exercise Sheet 2 – Solutions

Exercise 1: Consider the topological manifold $\mathbb{R}$ together with the two atlases $(\mathbb{R}, \mathrm{Id}_{\mathbb{R}})$ and $(\mathbb{R}, \psi)$, where $\psi\colon \mathbb{R} \to \mathbb{R},\ x \mapsto x^3$. Show that the corresponding smooth structures on $\mathbb{R}$ are different, but they are diffeomorphic to each other, i.e., there is a diffeomorphism $(\mathbb{R}, \mathrm{Id}_{\mathbb{R}}) \to (\mathbb{R}, \psi)$.

Solution: The union of the atlases $\mathcal{A} \coloneqq \{(\mathbb{R}, \mathrm{Id}_{\mathbb{R}})\}$ and $\mathcal{A}' \coloneqq \{(\mathbb{R}, \psi)\}$ is not a smooth atlas, because the transition map $\mathrm{Id}_{\mathbb{R}} \circ \psi^{-1} : y \mapsto y^{1/3}$ is not differentiable at the origin. Hence, these atlases determine different smooth structures on $\mathbb{R}$.

Consider the map

$$\begin{aligned} F\colon (\mathbb{R}, \mathcal{A}) &\to (\mathbb{R}, \mathcal{A}') \\ x &\mapsto x^{1/3}. \end{aligned}$$

The coordinate representation of this map is

$$\widehat{F}(t) = (\psi \circ F \circ \mathrm{Id}_{\mathbb{R}}^{-1})(t) = t,$$

which is smooth. The coordinate representation of its inverse is

$$\widehat{F^{-1}}(s) = (\mathrm{Id}_{\mathbb{R}} \circ F^{-1} \circ \psi^{-1})(s) = s,$$

which is smooth as well. Hence, F is a diffeomorphism.

Remark. In conclusion, we exhibited two distinct smooth structures on $\mathbb{R}$, but then proved that they are diffeomorphic. It is in fact true that any two smooth structures on $\mathbb{R}$ are diffeomorphic to each other, i.e., $\mathbb{R}$ admits a unique smooth structure up to diffeomorphism. However, there are topological manifolds admitting several smooth structures which are not diffeomorphic (google, for example, "exotic spheres").

It might seem confusing that the new structure we introduced (that is, endowing a topological manifold with a maximal smooth atlas) is not invariant under the natural notion of isomorphism (i.e., diffeomorphisms). The following analogy might clarify the situation:

Suppose our objects of study are sets and the functions between them. One might then be interested in endowing a set with a notion of symmetry, and so one is led to endowing a set with a group structure. Of course, we can endow the same set with different group structures. For example, a set with two elements $\{x, y\}$ can be endowed with a group structure where x is the neutral element, but it can also be endowed with a group structure where y is the neutral element. From the point of view of the original set $\{x, y\}$, these are different form each other. However, these two group structures are isomorphic to each other from the point of view of group theory. Furthermore, similarly to the situation in the exercise, $\{x, y\}$ admits a unique group structure up to isomorphism of groups. However, there are other sets like $\{x, y, z, w\}$ which admit several non-isomorphic group structures.

Exercise 2 (*Finite-dimensional vector spaces*)**:** Let V be an $\mathbb{R}$-vector space of dimension n. Recall that any norm on V determines a topology, which is independent of the choice of norm. Show that V has a natural smooth manifold structure as follows:

(a) Pick a basis $E_1, \dots, E_n$ for V and consider the map

$$E\colon \mathbb{R}^n \to V,\ (x^1, \dots, x^n) \mapsto \sum_{i=1}^{n} x^i E_i.$$

Show that (V, E^{-1}) is a chart for V; in particular, with the topology defined above, V is thus a topological n-manifold.

(b) Given a different basis $\widetilde{E}_1, \dots, \widetilde{E}_n$ for V, show that the charts (V, E^{-1}) and $(V, \widetilde{E}^{-1})$ are smoothly compatible. The collection of all such charts of V defines a smooth structure, called the *standard smooth structure on V*.

Solution: Denote by $\|\bullet\|_V : V \to \mathbb{R}_{\geq 0}$ the chosen norm on V, and by $\|\bullet\|_{\mathbb{R}^n}$ the standard Euclidean norm on $\mathbb{R}^n$.

(a) It suffices to show that E is a homeomorphism. First, observe that E is bijective. Now, note that the map $\|\cdot\|'\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ given by $\|x\|' := \|E(x)\|_V$ is a norm on $\mathbb{R}^n$. As all norms on $\mathbb{R}^n$ are equivalent, there exists a constant $c > 1$ such that

$$\frac{1}{c}\|x\|_{\mathbb{R}^n} \leq \|x\|' \leq c\|x\|_{\mathbb{R}^n} \quad \text{for all } x \in \mathbb{R}^n.$$

In particular, both E and E^{-1} are Lipschitz-continuous, and thus E is a homeomorphism.

(b) There exists an invertible matrix $A = (A_i^j)_{1\leq i,j\leq n} \in \mathrm{GL}(n, \mathbb{R})$ such that $E_i = \sum_j A_i^j \widetilde{E}_j$ for each i. Thus, the transition map $\widetilde{E}^{-1} \circ E$ is given by

$$\big(\widetilde{E}^{-1} \circ E\big)(x) = \widetilde{E}^{-1}\Big(\sum_i x^i E_i\Big) = \sum_i x^i \widetilde{E}^{-1}(E_i) = \sum_i x^i A_i^j.$$

Hence, the transition map is an invertible linear map, and thus a diffeomorphism (the partial derivatives of the first order are given by constant maps corresponding to the entries of A, and the partial derivatives of higher order vanish).

Exercise 3: Prove the following assertions:

(a) The space $M(m \times n, \mathbb{R})$ of $m \times n$ matrices with real entries has a natural smooth manifold structure.

(b) The *general linear group* $\mathrm{GL}(n, \mathbb{R})$ (i.e., the group of invertible $n \times n$ matrices with real entries) has a natural smooth manifold structure.

(c) The subset $M_m(m \times n, \mathbb{R})$ of $M(m \times n, \mathbb{R})$ of matrices of rank m, where $m < n$ has a natural smooth manifold structure. Similarly for $M_n(m \times n, \mathbb{R})$ when $n < m$.

(d) The space $\mathcal{L}(V, W)$ of $\mathbb{R}$-linear maps from V to W, where V and W are two finite-dimensional $\mathbb{R}$-vector spaces, has a natural smooth manifold structure.

What is the dimension of each of the above smooth manifolds?

Solution:

(a) The set $M(m \times n, \mathbb{R})$ is an $\mathbb{R}$-vector space of dimension mn, and thus by *Exercise* 2 it has a natural smooth structure, given by identifying it with $\mathbb{R}^{mn}$. We have

$$\dim M(m \times n, \mathbb{R}) = mn.$$

(b) Let $\det\colon M(n \times n, \mathbb{R}) \to \mathbb{R}$ be the determinant function. Note that it is continuous, and hence $\mathrm{GL}(n, \mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$ is an open subset of $M(n \times n, \mathbb{R})$. As the latter has a natural smooth manifold structure by (a), the open subset $\mathrm{GL}(n, \mathbb{R}) \subseteq M(n \times n, \mathbb{R})$ inherits a natural smooth manifold structure as well. We have

$$\dim \mathrm{GL}(n, \mathbb{R}) = n^2.$$

(c) By linear algebra we know that an $m \times n$-matrix with $m < n$ has full rank if and only if it has an invertible $m \times m$-submatrix. For a subset $I \subseteq \{1, \dots, n\}$ of cardinality m and a matrix $A \in M(m \times n, \mathbb{R})$, denote by A_I the $m \times m$ submatrix corresponding to the columns indexed by I. Consider the map

$$\begin{aligned} \det_I\colon M(m \times n, \mathbb{R}) &\to \mathbb{R} \\ A &\mapsto \det(A_I) \end{aligned}$$

and observe that it is continuous. Thus,

$$M_m(m \times n, \mathbb{R}) = \bigcup_{\substack{I \subseteq \{1,\dots,n\} \\ |I| = m}} \det_I^{-1}(\mathbb{R} \setminus \{0\})$$

is an open subset of $M(m \times n, \mathbb{R})$, and hence $M_m(m \times n, \mathbb{R})$ inherits a natural smooth manifold structure. We have

$$\dim M_m(m \times n, \mathbb{R}) = mn.$$

Finally, note that the isomorphism of vector spaces $M(m \times n, \mathbb{R}) \to M(n \times m, \mathbb{R})$ given by transposition preserves the rank. Therefore, $M_n(m \times n, \mathbb{R})$ with $n < m$ is again open in $M(m \times n, \mathbb{R})$, and hence inherits a natural smooth manifold structure.

(d) The set $\mathcal{L}(V, W)$ is naturally an $\mathbb{R}$-vector space, and hence it has a natural smooth manifold structure by *Exercise* 2. Indeed, fixing bases of V and W, $\mathcal{L}(V, W)$ can be naturally identified with $M(m \times n, \mathbb{R})$, where $m = \dim W$ and $n = \dim V$. We have

$$\dim \mathcal{L}(V, W) = \dim V \cdot \dim W.$$

Exercise 4 (*Product manifolds*)**:** Let $M_1, \dots, M_k$ be smooth manifolds of dimensions $n_1, \dots, n_k$, respectively, where $k \geq 2$. Show that the product space $M_1 \times \ldots \times M_k$ is a smooth manifold of dimension $n_1 + \ldots + n_k$ by constructing a smooth manifold structure on it.

Solution: By [*Exercise Sheet* 1*, Exercise* 5] we know that $M_1 \times \ldots \times M_k$ is a topological manifold of dimension $n_1 + \cdots + n_k$. As in the solution of [*Exercise Sheet* 1*, Exercise* 5], we see that if $\mathcal{A}_i$ denotes the smooth structure of M_i for $1 \leq i \leq k$, then

$$\mathcal{A} \coloneqq \big\{ (U_1 \times \cdots \times U_k, \varphi_1 \times \cdots \times \varphi_k) \mid (U_1, \varphi_1) \in \mathcal{A}_1, \dots, (U_k, \varphi_k) \in \mathcal{A}_k \big\}$$

is an atlas for $M_1 \times \ldots \times M_k$. To see that it is smooth, observe that the transition map between two charts $(U_1 \times \cdots \times U_k, \varphi_1 \times \cdots \times \varphi_k)$ and $(V_1 \times \cdots \times V_k, \psi_1 \times \cdots \times \psi_k)$ of $\mathcal{A}$ is given by

$$(\psi_1 \circ \varphi_1^{-1}) \times \ldots \times (\psi_k \circ \varphi_k^{-1}),$$

which is smooth, since each factor is smooth. Therefore, $\mathcal{A}$ is a smooth atlas, and hence determines a smooth structure on $M_1 \times \ldots \times M_k$ by *Proposition 1.8.*

Exercise 5: Consider the n-sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$. Denote by $N = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$ the *north pole* and by $S = -N = (0, \dots, 0, -1)$ the *south pole* of $\mathbb{S}^n$. Define the *stereographic projection from the north pole* N as follows:

$$\sigma \colon \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n, \quad \sigma(x^1, \dots, x^{n+1}) = \frac{1}{1 - x^{n+1}} \, (x^1, \dots, x^n).$$

Let $\widetilde{\sigma}(x) = -\sigma(-x)$ for $x \in \mathbb{S}^n \setminus \{S\}$; it is called the *stereographic projection from the south pole.*

(a) For any $x \in \mathbb{S}^n \setminus \{N\}$, show that $\sigma(x) = u$, where $(u, 0)$ is the point where the line through N and x intersects the linear subspace where $x^{n+1} = 0$. Similarly, show that $\widetilde{\sigma}(x)$ is the point where the line through S and x intersects the same subspace.

(b) Show that σ is bijective, and

$$\sigma^{-1}(u^1, \dots, u^n) = \frac{1}{|u|^2 + 1} \, (2u^1, \dots, 2u^n, |u|^2 - 1).$$

(c) Verify that the atlas consisting of the two charts $\big(\mathbb{S}^n \setminus \{N\}, \sigma\big)$ and $\big(\mathbb{S}^n \setminus \{S\}, \widetilde{\sigma}\big)$ is a smooth atlas for $\mathbb{S}^n$, and hence defines a smooth structure on $\mathbb{S}^n$. (The coordinates defined by σ or $\widetilde{\sigma}$ are called *stereographic coordinates.*)

(d) Show that the smooth structure determined by the above atlas is the same as the one defined via graph coordinates in the lecture.

Solution: Denote by H the linear subspace of $\mathbb{R}^{n+1}$ where $x^{n+1} = 0$, i.e.,

$$H = \{(x^1, \dots, x^{n+1}) \in \mathbb{R}^{n+1} \mid x^{n+1} = 0\},$$

and observe that H can be identified with $\mathbb{R}^n$.

(a) The line $\ell_1 \subseteq \mathbb{R}^{n+1}$ through $N = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$ and $x = (x^1, \dots, x^{n+1}) \in \mathbb{S}^n \backslash \{N\}$ is given by the parametric equation

$$\ell_1 : (x^1, \dots, x^{n+1}) + t(-x^1, \dots, -x^n, 1 - x^{n+1}),\ t \in \mathbb{R}$$

and intersects the hyperplane $H : (x^{n+1} = 0)$ for $t = -\frac{x^{n+1}}{1-x^{n+1}}$ at the point $(u, 0) \in \mathbb{R}^{n+1}$, where

$$\begin{aligned} u &= \left(x^1 + \frac{x^{n+1}}{1 - x^{n+1}} x^1, \dots, x^n + \frac{x^{n+1}}{1 - x^{n+1}} x^n\right) \\ &= \left(\frac{x^1}{1 - x^{n+1}}, \dots, \frac{x^n}{1 - x^{n+1}}\right) = \sigma(x) \in \mathbb{R}^n. \end{aligned}$$

Similarly, we see that

$$\begin{aligned} \widetilde{\sigma}(x) = -\sigma(-x) &= -\left(-\frac{x^1}{1 + x^{n+1}}, \dots, -\frac{x^n}{1 + x^{n+1}}\right) \\ &= \left(\frac{x^1}{1 + x^{n+1}}, \dots, \frac{x^n}{1 + x^{n+1}}\right) \end{aligned}$$

is the point where the line $\ell_2 \subseteq \mathbb{R}^{n+1}$ through S and x intersects the hyperplane H.

(b) Pick a point $u = (u^1, \dots, u^n) \in \mathbb{R}^n$. The line $\ell \subseteq \mathbb{R}^{n+1}$ through $(u, 0) \in \mathbb{R}^{n+1}$ and $N = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$ is given by the parametric equation

$$\ell : (u^1, \dots, u^n, 0) + t(-u^1, \dots, -u^n, 1),\ t \in \mathbb{R}$$

and intersects the n-sphere $\mathbb{S}^n$ at points which satisfy the equation

$$|u|^2(1 - t)^2 + t^2 = 1,$$

where $|u|^2 = \sum_{i=1}^n (u^i)^2$. It is now easy to check that the above equation has two solutions: $t = 1$, which corresponds to the point $N \in \mathbb{S}^n$, and $t = \frac{|u|^2-1}{|u|^2+1} \neq 1$, which corresponds to the point

$$x = \left(\frac{2u^1}{|u|^2 + 1}, \dots, \frac{2u^n}{|u|^2 + 1}, \frac{|u|^2 - 1}{|u|^2 + 1}\right) \in \mathbb{S}^n.$$

Therefore, the map

$$\sigma\colon \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n, \quad (x^1, \dots, x^{n+1}) \mapsto \frac{1}{1 - x^{n+1}}\,(x^1, \dots, x^n)$$

is bijective, and its inverse $\sigma^{-1}\colon \mathbb{R}^n \to \mathbb{S}^n \setminus \{N\}$ is given by the formula

$$(u^1, \ldots, u^n) \mapsto \frac{1}{|u|^2 + 1}\,(2u^1, \ldots, 2u^n, |u|^2 - 1).$$

(c) It is straightforward to check that

$$\big(\widetilde{\sigma} \circ \sigma^{-1}\big)(u^1, \ldots, u^n) = \frac{1}{|u|^2}(u^1, \ldots, u^n),\ (u^1, \ldots, u^n) \in \mathbb{R}^n \setminus \{(0, \ldots, 0)\}$$

and that its inverse $\sigma \circ \widetilde{\sigma}^{-1}$ is also given by the same formula, namely,

$$\big(\sigma \circ \widetilde{\sigma}^{-1}\big)(u^1, \ldots, u^n) = \frac{1}{|u|^2}(u^1, \ldots, u^n),\ (u^1, \ldots, u^n) \in \mathbb{R}^n \setminus \{(0, \ldots, 0)\}.$$

Since both $\widetilde{\sigma} \circ \sigma^{-1}$ and $\sigma \circ \widetilde{\sigma}^{-1}$ are clearly smooth, the two charts $\big(\mathbb{S}^n \setminus \{N\}, \sigma\big)$ and $\big(\mathbb{S}^n \setminus \{S\}, \widetilde{\sigma}\big)$ for $\mathbb{S}^n$ are smoothly compatible, and since their domains clearly cover $\mathbb{S}^n$, they comprise a smooth atlas for $\mathbb{S}^n$, which determines a smooth structure on $\mathbb{S}^n$ by *Proposition 1.8*(a).

(d) According to *Proposition 1.8*(b), to prove the claim, we have to check that the graph coordinates $\big(U_i^\pm \cap \mathbb{S}^n, \varphi_i^\pm\big)$, where

$$\varphi_i^\pm\colon U_i^\pm \cap \mathbb{S}^n \to \mathbb{B}^n,\ (x^1, \ldots, x^{n+1}) \mapsto (x^1, \ldots, \widehat{x^i}, \ldots x^{n+1})$$

with inverse

$$\big(\varphi_i^\pm\big)^{-1}\colon \mathbb{B}^n \to U_i^\pm \cap \mathbb{S}^n,\ (u^1, \ldots, u^n) \mapsto \big(u^1, \ldots, \pm\sqrt{1 - |u|^2}, \ldots, u^n\big),$$

and the stereographic coordinates $\big(\mathbb{S}^n \setminus \{N\}, \sigma\big)$ and $\big(\mathbb{S}^n \setminus \{S\}, \widetilde{\sigma}\big)$ are smoothly compatible. For instance, we have

$$\big(\varphi_i^\pm \circ \sigma^{-1}\big)(u^1, \ldots, u^n) = \left(\frac{2u^1}{1 + |u|^2}, \ldots, \widehat{\frac{2u^i}{1 + |u|^2}}, \ldots, \frac{2u^n}{1 + |u|^2}, \frac{|u|^2 - 1}{1 + |u|^2}\right)$$

for $1 \le i \le n$, and

$$\big(\varphi_{n+1}^\pm \circ \sigma^{-1}\big)(u^1, \ldots, u^n) = \left(\frac{2u^1}{1 + |u|^2}, \ldots, \frac{2u^i}{1 + |u|^2}, \ldots, \frac{2u^n}{1 + |u|^2}\right),$$

which are clearly smooth in their domain of definition. In a similar fashion one can readily verify that the remaining charts are smoothly compatible; this yields the assertion.