Differential Geometry II - Smooth Manifolds
Winter Term 2024/2025
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Exercise Sheet 1

Exercise 1:
Show that if a topological space M is locally Euclidean at some point $p \in M$ (i.e., p has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$), then p has a neighborhood that is homeomorphic to the whole space $\mathbb{R}^n$ or to an open ball in $\mathbb{R}^n$.

Exercise 2:
Examine which of the following spaces (endowed with the subspace topology) is locally Euclidean:

(a) The closed interval $[0, 1] \subseteq \mathbb{R}$.

(b) The "bent line" $\big\{(x, y) \in \mathbb{R}^2 \mid x \geq 0,\ y \geq 0,\ xy = 0\big\} \subseteq \mathbb{R}^2$.

Exercise 3:

(a) *The line with two origins*: Consider the set

$$X = \big\{(x, y) \in \mathbb{R}^2 \mid y \in \{-1, 1\}\big\} \subseteq \mathbb{R}^2$$

and let M be the quotient of X by the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Show that M is locally Euclidean and second-countable, but not Hausdorff.

(b) Show that a disjoint union of uncountably many copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second-countable.

Exercise 4 (to be submitted by Thursday, 19.09.2024, 16:00):
Consider the subset

$$V = \big\{(x, y) \in \mathbb{R}^2 \mid (x - 1)(x - y) = 0\big\} \subseteq \mathbb{R}^2$$

endowed with the subspace topology. Show that V is not a topological manifold.

Exercise 5 (*Product manifolds*)**:**
Let $M_1, \ldots, M_k$ be topological manifolds of dimensions $n_1, \ldots, n_k$, respectively, where $k \geq 2$. Show that the product space $M_1 \times \ldots \times M_k$ is a topological manifold of dimension $n_1 + \ldots + n_k$.

Remark. The *n-torus*

$$\mathbb{T}^n \coloneqq \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$$

is a topological n-manifold by *Exercise* 5.