ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Quantum Computation — Assignment date: July 8th, 2022, 15:15
Spring 2022 — Due date: July 8th, 2022, 18:15

CS 308 – Final Exam – room AAC 231

There are 4 problems. Use scratch paper if needed to figure out the solution. Write your final answer in the indicated space. A cheat sheet with 2 A4 pages is allowed. No electronic devices allowed. Good luck!

Name: ______________________________

Section: _______________

Sciper No.: _______________

Problem 1	/ 6
Problem 2	/ 8
Problem 3	/ 16
Problem 4	/ 12
Total	/42

Problem 1. (6 pts)

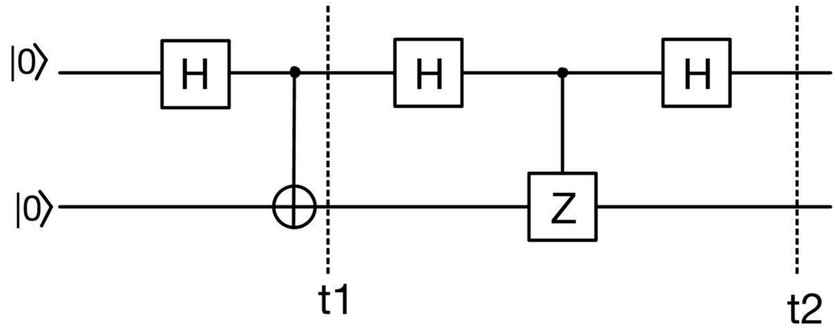


Figure 1: A simple quantum circuit

Consider the quantum circuit in Figure 1,

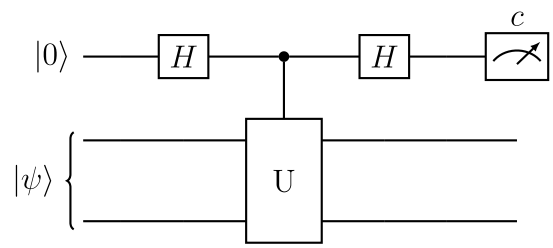
1. (1pt) Calculate the state at t_1.
2. (2pt) Calculate the state at t_2.
3. (1pt) Is the state entangled at t_1?
4. (1pt) Is the state entangled at t_2?
5. (1pt) Suggest a simpler circuit (with less gates) that prepares the same state at t_2 from the same starting state $|0\rangle \otimes |0\rangle$.

Solution to problem 1:

1. (1pt) $|\psi\rangle_{t_1} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle$.
2. (1pt) $|\psi\rangle_{t_2} = |0\rangle \otimes (\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.
3. (1pt) Yes, $|\psi\rangle_{t_1}$ is the maximally entangled state and cannot be expressed as the tensor product of two states.
4. (1pt) No, it is a product state.
5. (1pt) A circuit which applies the Hadamard gate to the second qubit will produce the same state at t_2 when initialized with $|0\rangle \otimes |0\rangle$.

Problem 2. *The U-Test* (8 pts)

1. (4 pts) Consider the "control-U test" for some unitary operator $U : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$:



   Given the initial state $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ and a measurement stored in a classical bit c, show that we have the following probability:

$$p_0 = \mathcal{P}(c = 0) = \frac{1}{2}\left(1 + \Re(\langle\psi| U |\psi\rangle)\right)$$

2. (2 pts) Say U is the swap operator such that for any $|u\rangle \otimes |v\rangle \in \mathbb{C}^2 \times \mathbb{C}^2$, we have $U |u\rangle \otimes |v\rangle = |v\rangle \otimes |u\rangle$.

   (a) What is p_0 for $|\psi\rangle = |u\rangle \otimes |v\rangle$? Give the solution in terms of $\langle u|v\rangle$.

   (b) What is p_0 for $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$? (Give the numerical value of p_0)

3. (2 pts) Say $U = H \otimes H$.

   (a) What is p_0 for $|\psi\rangle = |u\rangle \otimes |v\rangle$? (Give the numerical value of p_0)

   (b) What is p_0 for $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$? (Give the numerical value of p_0)

Solution to problem 2:

1.

$$\psi_1 = (H\otimes I)CU((H|0\rangle)\otimes|\psi\rangle) = \frac{1}{\sqrt{2}}(H\otimes I)CU(|0\rangle\otimes|\psi\rangle + |1\rangle\otimes|\psi\rangle) \quad (1)$$
$$= \frac{1}{\sqrt{2}}(H\otimes I)(|0\rangle\otimes|\psi\rangle + |1\rangle\otimes U|\psi\rangle) \quad (2)$$
$$= \frac{1}{2}(|0\rangle + |1\rangle)\otimes|\psi\rangle + \frac{1}{2}(|0\rangle - |1\rangle)\otimes U|\psi\rangle \quad (3)$$
$$= \frac{1}{2}|0\rangle\otimes(|\psi\rangle + U|\psi\rangle) + \frac{1}{2}|1\rangle\otimes(|\psi\rangle - U|\psi\rangle) \quad (4)$$

So:

$$p_0 = \langle\psi_1|(|0\rangle\langle 0|\otimes I)|\psi_1\rangle \quad (5)$$
$$= \frac{1}{4}(\langle\psi| + \langle\psi|U^\dagger)(|\psi\rangle + U|\psi\rangle) \quad (6)$$
$$= \frac{1}{4}(\langle\psi|\psi\rangle + \langle\psi|U|\psi\rangle + \langle\psi|U^\dagger|\psi\rangle + \langle\psi|U^\dagger U|\psi\rangle) \quad (7)$$
$$= \frac{1}{2}(1 + \Re(\langle\psi|U|\psi\rangle)) \quad (8)$$

2. for the swap operator:

   (a) $U|u\rangle\otimes|v\rangle = |v\rangle\otimes|u\rangle$ so:

   $$p_0 = \frac{1}{2}\left(1 + \|\langle u|v\rangle\|^2\right)$$

   (b) for the bell state, $U|\psi\rangle = |\psi\rangle$ so $p_0 = 1$.

3. for the double Hadamard gate:

   (a) if $|\psi\rangle = |u\rangle\otimes|v\rangle$ for $u, v \in \{0,1\}$, then $(H\otimes H)|\psi\rangle = \frac{1}{2}(|0\rangle + (-1)^u|1\rangle)\otimes(|0\rangle + (-1)^v|1\rangle)$ so:

   $$p_0 \in \left\{\frac{3}{4}, \frac{1}{4}\right\}$$

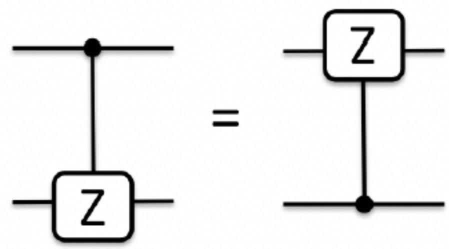
   (b) for the bell state, we find:

   $$(H\otimes H)|\psi\rangle = \frac{1}{2\sqrt{2}}\left[(|0\rangle + |1\rangle)\otimes(|0\rangle + |1\rangle) + (|0\rangle - |1\rangle)\otimes(|0\rangle - |1\rangle)\right]$$
   $$= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\psi\rangle$$

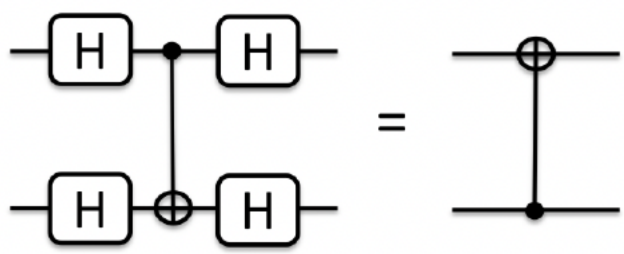
   So we find again $p_0 = 1$

Problem 3. *Circuit identities* (16 pts).

1. (4pts) Show that:



2. (4pts) Show that:



3. (4pts) Let $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and CNOT the control-not gate with the first qubit as the control bit and the second as the target bit. Consider the identity:

$$\text{CNOT}(X \otimes I)\text{CNOT} = X \otimes X$$

As above, draw the equality in terms of circuits. Then, prove the identity.

4. (4pts) Let $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and CNOT the control-not gate with the first qubit as the control bit and the second as the target bit. Consider the identity:

$$\text{CNOT}(Z \otimes I)\text{CNOT} = Z \otimes I$$

Draw the equality in terms of circuits. Prove the identity.

Solution to problem 3:

1. We check the identity on computational basis states. By linearity its then true in general. If x is the control bit:

$$CZ|x,y\rangle = |x\rangle \otimes Z^x|y\rangle = |x\rangle \otimes (-1)^{yx}|y\rangle = (-1)^{xy}|x,y\rangle$$

   If y is the control bit

$$CZ|x,y\rangle = Z^y|x,y\rangle = (-1)^{xy}|x,y\rangle$$

   so we have equality.

2. We have

$$\begin{aligned} H \otimes H|x,y\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + (-1)^y|1\rangle) \\ &= \frac{1}{2}(|00\rangle + (-1)^y|01\rangle + (-1)^x|10\rangle + (-1)^x(-1)^y|11\rangle) \end{aligned}$$

   Then

$$\text{CNOT}(H \otimes H)|x,y\rangle = \frac{1}{2}(|00\rangle + (-1)^y|01\rangle + (-1)^x|11\rangle + (-1)^x(-1)^y|10\rangle)$$

   Now lets compute

$$\begin{aligned} (H \otimes H)|x \oplus y, y\rangle &= \frac{1}{2}(|0\rangle + (-1)^{x\oplus y}|1\rangle) \otimes (|0\rangle + (-1)^y|1\rangle) \\ &= \frac{1}{2}(|00\rangle + (-1)^y|01\rangle + (-1)^x|10\rangle + (-1)^x(-1)^y|11\rangle) \end{aligned}$$

   Putting together the last two results we find

$$(H \otimes H)\text{CNOT}(H \otimes H)|x,y\rangle = |x \oplus y, y\rangle$$

   which is the asked identity.

3. We denote by $\bar{x}$ and $\bar{y}$ the negations of x and y. We have

$$\text{CNOT}(X\otimes I)\text{CNOT}|x,y\rangle = \text{CNOT}(X\otimes I)|x,y\oplus x\rangle = \text{CNOT}|\bar{x},y\oplus x\rangle = |\bar{x},\bar{y}\rangle = X\otimes X|xy\rangle$$

   where we used in the next-to-last equality $y \oplus x \oplus \bar{x} = y \oplus 1 = \bar{y}$.

4. For the last identity we have

$$\text{CNOT}(Z\otimes I)\text{CNOT}|x,y\rangle = \text{CNOT}(Z\otimes I)|x,y\oplus x\rangle = (-1)^x\text{CNOT}|x,y\oplus x\rangle = (Z\otimes I)|x,y\rangle$$

Problem 4. *Hamiltonian simulation* (12 pts)

For any $N \times N$ matrix M the exponential is defined as

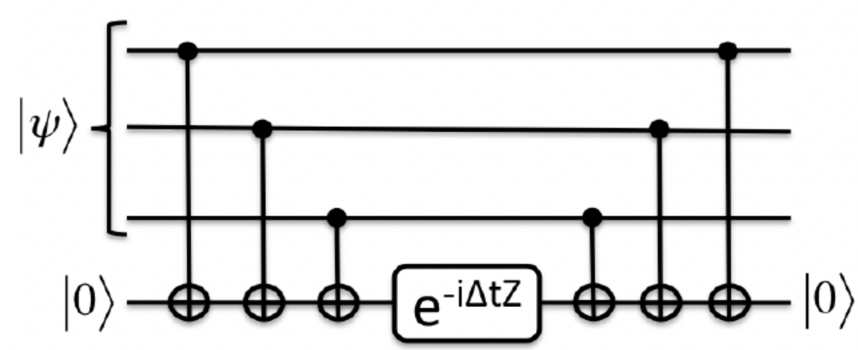
$$e^M = \sum_{k=0}^{\infty} \frac{M^k}{k!} .$$

We admit the following two properties (without proof):

1. If λ is an eigenvalue of M then e^λ is an eigenvalue of e^M.
2. If U is a unitary $N \times N$ matrix then $Ue^M U^\dagger = e^{UMU^\dagger}$.

Let $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and Δt is a real number. Let $A = X \otimes X \otimes X$. We want to construct a circuit that computes $e^{-i\Delta t A}|\psi\rangle$ where $|\psi\rangle \in (\mathbb{C}^2)^{\otimes 3}$ is given.

1. (2pts) Find a unitary matrix U such that $(U \otimes U \otimes U)A(U^\dagger \otimes U^\dagger \otimes U^\dagger) = A'$ where $A' = Z \otimes Z \otimes Z$.
2. (6pts) Show that the following circuit computes $e^{-i\Delta t A'}|\psi\rangle \otimes |0\rangle$.
   Hint: First take a computational basis state $|\psi\rangle = |x, y, z\rangle$ where $x, y, z \in \{0, 1\}$. Then justify the statement for general $|\psi\rangle$.



3. (4pts) Deduce and draw a circuit that computes $e^{-i\Delta t A}|\psi\rangle \otimes |0\rangle$ when $|\psi\rangle \otimes |0\rangle$ is the input.
4. (Bonus question 2pts) Prove the preliminary statements (1) and (2) at the beginning of the problem.

Solution to problem 4:

1. We know (or can check) that $HXH = Z$ where H is the Hadamard matrix. Thus $(H \otimes H \otimes H)A(H \otimes H \otimes H) = A'$. Since H is unitary and with $H = H^\dagger$ we have that $U = H$.

2. Since any state can be decomposed on the computational basis $|\psi\rangle = \sum_{x,y,z} C_{xyz}|xyz\rangle$ by linearity it suffices compute the output for a computational basis state $|\psi\rangle = |xyz\rangle$. After the first series of control-not gates we have the state $|xyz\rangle \otimes |x \oplus y \oplus z\rangle$. Now we act with $e^{i\Delta t Z}$ on the last qubit. This gives

$$|xyz\rangle \otimes e^{i\Delta t Z}|x \oplus y \oplus z\rangle = |xyz\rangle \otimes e^{i\Delta t(-1)^{x\oplus y\oplus z}}|x \oplus y \oplus z\rangle \tag{9}$$

$$= e^{-i\Delta t(-1)^x(-1)^y(-1)^z}|xyz\rangle \otimes |x \oplus y \oplus z\rangle \tag{10}$$

Now we act with the last three control-not gates and find the output state

$$e^{-i\Delta t(-1)^x(-1)^y(-1)^z}|xyz\rangle \otimes 0\rangle$$

This is precisely equal to

$$e^{-i\Delta t Z\otimes Z\otimes Z}|xyz\rangle \otimes 0\rangle$$

3. We have

$$e^{-i\Delta t A}|\psi\rangle \otimes |0\rangle = (H \otimes H \otimes H)e^{-i\Delta t A'}(H \otimes H \otimes H)|\psi\rangle \otimes |0\rangle$$

Thus the circuit is obtained by appending H gates at the beginning and end of the first three lines (picture).

4. Bonus question (2pts). For the first statement we first notice that $M|\phi\rangle = \lambda|\phi\rangle$ implies $M^k|\phi\rangle = \lambda|\phi\rangle$. Then act on $|\phi\rangle$ with the expansion of the exponential, and resum, to get the result.

For the second statement, expanding the exponential we have

$$(U \otimes U \otimes U)e^{-i\Delta t A}(U^\dagger \otimes U^\dagger \otimes U^\dagger) = \sum_{n=1}^{\infty} \frac{(-i\Delta t)^n}{n!}(U \otimes U \otimes U)A^n(U^\dagger \otimes U^\dagger \otimes U^\dagger) \tag{11}$$

$$= \sum_{n=1}^{\infty} \frac{(-i\Delta t)^n}{n!}A'^n \tag{12}$$

$$= e^{i\Delta t A'} \tag{13}$$

where we used

$$(U\otimes U\otimes U)A^2(U^\dagger\otimes U^\dagger\otimes U^\dagger) = (U\otimes U\otimes U)A(U^\dagger\otimes U^\dagger\otimes U^\dagger)(U\otimes U\otimes U)A(U^\dagger\otimes U^\dagger\otimes U^\dagger) = A'^2$$

by unitarity of U, and similarly for $n > 2$.