Astrophysics IV: Stellar and galactic dynamics

Solutions

Problem 1:

As the system is spherical, the distribution function (DF) depends on the energy and the norm of the total angular momentum. In spherical coordinates, the Hamiltonian writes:

$$H(\mathbf{x},\mathbf{v}) = \frac{1}{2}v_r^2 + \frac{1}{2}v_\phi^2 + \frac{1}{2}v_\theta^2 + \Phi(r), \tag{1}$$

and the norm of the total angular momentum is :

$$L(\mathbf{x},\mathbf{v}) = rv_t = \sqrt{v_\phi^2 + v_\theta^2}, \tag{2}$$

where v_t is the norm of the tangential velocity. The DF write:

$$f\left(\frac{1}{2}v_r^2 + \frac{1}{2}v_\phi^2 + \frac{1}{2}v_\theta^2 + \Phi(r), r\sqrt{v_\phi^2 + v_\theta^2}\right). \tag{3}$$

By definition, the mean radial velocity is:

$$\bar{v}_r = \frac{1}{\nu(\mathbf{x})}\int_{-\infty}^{\infty} dv_\phi \int_{-\infty}^{\infty} dv_\theta \int_{-\infty}^{\infty} dv_r v_r f\left(\frac{1}{2}v_r^2 + \frac{1}{2}v_\phi^2 + \frac{1}{2}v_\theta^2 + \Phi(r), r\sqrt{v_\phi^2 + v_\theta^2}\right). \tag{4}$$

As the integrant is odd in v_r, the integral is 0 and thus $\bar{v}_r = 0$. The mean velocity in θ is:

$$\bar{v}_\theta = \frac{1}{\nu(\mathbf{x})}\int_{-\infty}^{\infty} dv_\phi \int_{-\infty}^{\infty} dv_r \int_{-\infty}^{\infty} dv_\theta v_\theta f\left(\frac{1}{2}v_r^2 + \frac{1}{2}v_\phi^2 + \frac{1}{2}v_\theta^2 + \Phi(r), r\sqrt{v_\phi^2 + v_\theta^2}\right). \tag{5}$$

As the integrant is odd in v_θ, the integral is 0 and thus $\bar{v}_\theta = 0$. Similarly we will get $\bar{v}_\phi = 0$ and thus $\bar{v}_t = 0$.

By definition, σ_r^2 is:

$$\sigma_r^2 = \frac{1}{\nu(\mathbf{x})}\int_{-\infty}^{\infty} dv_\phi \int_{-\infty}^{\infty} dv_\theta \int_{-\infty}^{\infty} dv_r v_r^2 f\left(\frac{1}{2}v_r^2 + \frac{1}{2}v_\phi^2 + \frac{1}{2}v_\theta^2 + \Phi(r), r\sqrt{v_\phi^2 + v_\theta^2}\right). \tag{6}$$

As the integrant is even in v_r, $\sigma_r^2 \neq 0$. To express the latter in term of v_t, we use:

$$v_\phi = v_t\cos(\theta) \qquad \text{and} \qquad v_\theta = v_t\sin(\theta), \tag{7}$$

and deduce that $dv_\phi dv_\theta = v_t dv_t d_\theta$. With this change of coordinates, the previous integral writes:

$$\begin{aligned} \sigma_r^2 &= \frac{1}{\nu(\mathbf{x})}\int_0^\infty dv_t \int_0^{2\pi} d\theta \int_{-\infty}^\infty dv_r v_r^2 f\left(\frac{1}{2}v_r^2+\frac{1}{2}v_t^2+\Phi(r), rv_t\right). &(8)\\ &= \frac{2\pi}{\nu(\mathbf{x})}\int_{-\infty}^\infty dv_r v_r^2 \int_0^\infty dv_t v_t f\left(\frac{1}{2}v_r^2+\frac{1}{2}v_t^2+\Phi(r), rv_t\right). &(9)\end{aligned}$$

By definition, σ_ϕ^2 is:

$$\sigma_\phi^2 = \frac{1}{\nu(\mathbf{x})}\int_{-\infty}^\infty dv_r \int_{-\infty}^\infty dv_\theta \int_{-\infty}^\infty dv_\phi v_\phi^2 f\left(\frac{1}{2}v_r^2+\frac{1}{2}v_\phi^2+\frac{1}{2}v_\theta^2+\Phi(r), r\sqrt{v_\phi^2+v_\theta^2}\right). \tag{10}$$

With the same change of coordinates this becomes:

$$\begin{aligned} \sigma_\phi^2 &= \frac{1}{\nu(\mathbf{x})}\int_0^{2\pi} d\theta \int_{-\infty}^\infty dv_r \int_0^\infty dv_t v_t^2 v_t \cos(\theta) f\left(\frac{1}{2}v_r^2+\frac{1}{2}v_t^2+\Phi(r), rv_t\right). &(11)\\ &= \frac{\pi}{\nu(\mathbf{x})}\int_{-\infty}^\infty dv_r \int_0^\infty dv_t v_t^3 f\left(\frac{1}{2}v_r^2+\frac{1}{2}v_t^2+\Phi(r), rv_t\right). &(12)\end{aligned}$$

As the dependency of the DF on v_ϕ is exactly the same than the one on v_θ, we deduce that $\sigma_\theta^2 = \sigma_\phi^2$. Finally, it is easy to see that any other component of the velocity dispersion tensor will be zero ans the integrant of the integral will always be odd.

Problem 2:

As the system is axi-symmetric, the distribution function (DF) depends on the energy and the norm of the total angular momentum. In cylindrical coordinates, the Hamiltonian writes:

$$H(\mathbf{x},\mathbf{v}) = \frac{1}{2}v_R^2+\frac{1}{2}v_\phi^2+\frac{1}{2}v_z^2+\Phi(R,z), \tag{13}$$

and z-component of the angular momentum is :

$$L_z(\mathbf{x},\mathbf{v}) = R^2\dot{\phi} = Rv_\phi, \tag{14}$$

where v_ϕ is the norm of the azimuthal velocity. The DF write:

$$f\left(\frac{1}{2}v_R^2+\frac{1}{2}v_\phi^2+\frac{1}{2}v_z^2+\Phi(R,z), Rv_\phi\right). \tag{15}$$

By definition, the mean radial velocity is:

$$\bar{v}_R = \frac{1}{\nu(\mathbf{x})}\int_{-\infty}^\infty dv_\phi \int_{-\infty}^\infty dv_z \int_{-\infty}^\infty dv_R v_R f\left(\frac{1}{2}v_R^2+\frac{1}{2}v_\phi^2+\frac{1}{2}v_z^2+\Phi(R,z), Rv_\phi\right). \tag{16}$$

As the integrant is odd in v_R, the integral is 0 and thus $\bar{v}_R = 0$. The same conclusion hold true for $\bar{v}_z$, i.e., $\bar{v}_z = 0$ as the integrant as v_z may be permuted with v_R. The mean azimuthal velocity is:

$$\bar{v}_\phi = \frac{1}{\nu(\mathbf{x})}\int_{-\infty}^\infty dv_z \int_{-\infty}^\infty dv_R \int_{-\infty}^\infty dv_\phi v_\phi f\left(\frac{1}{2}v_R^2+\frac{1}{2}v_\phi^2+\frac{1}{2}v_z^2+\Phi(R,z), Rv_\phi\right). \tag{17}$$

Here as the DF depends on Rv_ϕ, the integrant is no longer odd and in the general case $\bar{v}_\phi \neq 0$.

By definition, σ_R^2 is:

$$\sigma_R^2 = \frac{1}{\nu(\mathbf{x})} \int_{-\infty}^{\infty} dv_\phi \int_{-\infty}^{\infty} dv_z \int_{-\infty}^{\infty} dv_R v_R^2 f\left(\frac{1}{2}v_R^2 + \frac{1}{2}v_\phi^2 + \frac{1}{2}v_z^2 + \Phi(R,z), Rv_\phi\right). \quad (18)$$

The integrant being even, in the general case we will have $\sigma_R^2 \neq 0$. Its easy to show that $\sigma_z^2 = \sigma_R^2$ as both quantities can be permuted without changing the value of the integral. The mean azimuthal velocity dispersion writes:

$$\sigma_\phi^2 = \frac{1}{\nu(\mathbf{x})} \int_{-\infty}^{\infty} dv_z \int_{-\infty}^{\infty} dv_R \int_{-\infty}^{\infty} dv_\phi (v_\phi - \bar{v}_\phi)^2 f\left(\frac{1}{2}v_R^2 + \frac{1}{2}v_\phi^2 + \frac{1}{2}v_z^2 + \Phi(R,z), Rv_\phi\right), \quad (19)$$

and in general is different than zero as well as different than σ_R^2 and σ_z^2.

Finally, it is obvious to see that any other component of the velocity dispersion tensor will be equal to zero.

Problem 3:

The anisotropic parameter is defined by:

$$\beta = 1 - \frac{\sigma_t^2}{2\sigma_r^2} \quad (20)$$

The radial velocity dispersion writes:

$$\sigma_r^2 = \frac{1}{\nu(\mathbf{x})} \int_{-\infty}^{\infty} dv_\phi \int_{-\infty}^{\infty} dv_\theta \int_{-\infty}^{\infty} dv_r v_r^2 f_1\left(\frac{1}{2}v_r^2 + \frac{1}{2}v_\phi^2 + \frac{1}{2}v_\theta^2 + \Phi(r)\right) r^\gamma \left(v_\phi^2 + v_\theta^2\right)^{\gamma/2}. \quad (21)$$

We perform the following change of variables (polar coordinates in velocity space):

$$\begin{cases} v_r = v\cos(\eta) \\ v_\theta = v\sin(\eta)\cos(\phi) \\ v_\phi = v\sin(\eta)\sin(\phi), \end{cases} \quad (22)$$

with $\eta \in [0,\pi]$ and $\phi \in [0, 2\pi]$. With these new variables, we get:

$$\begin{cases} v_r = v\cos(\eta) \\ v_t = v\sin(\eta) \\ L = r\sqrt{v_\phi^2 + v_\theta^2} = rv_t = rv\sin(\eta) \end{cases} \quad (23)$$

and

$$dv_r dv_\theta dv_\phi = v^2 \sin(\eta) dv \quad (24)$$

Any integral on the velocity space of the distribution function becomes:

$$\int_{-\infty}^{\infty} dv_\phi \int_{-\infty}^{\infty} dv_\theta \int_{-\infty}^{\infty} dv_r f(v_r, v_\phi, v_\theta) = 2\pi \int_0^\pi d\eta \sin(\eta) \int_0^\infty dv v^2 f\left(\frac{1}{2}v^2 + \Phi(r)\right). \quad (25)$$

The next step is to make the distribution function a function of the relative energy instead of the energy.

$$f\left(\frac{1}{2}v^2 + \Phi(r)\right) \to f\left(\psi(r) - \frac{1}{2}v^2\right) = f(\epsilon) \tag{26}$$

As $f(\epsilon) = 0$ for $\epsilon < 0$, this limits the maximal velocity to $\sqrt{2\psi(r)}$ and the previous integral becomes:

$$2\pi \int_0^\pi d\eta \sin(\eta) \int_0^{\sqrt{2\psi(r)}} dv v^2 f\left(\frac{1}{2}v^2 + \Phi(r)\right). \tag{27}$$

Back to the radial velocity dispersion, using Eq. 23 we can rewrite Eq. 21 as:

$$\sigma_r^2 = 2\pi r^\gamma \int_0^\pi d\eta \sin^{\gamma+1}(\eta)\cos^2(\eta) \int_0^{\sqrt{2\psi(r)}} dv v^{\gamma+2} f_1(\epsilon) \tag{28}$$

Similarly, we can express the tangential velocity dispersion as:

$$\sigma_t^2 = 2\pi r^\gamma \int_0^\pi d\eta \sin^{\gamma+3}(\eta) \int_0^{\sqrt{2\psi(r)}} dv v^{\gamma+2} f_1(\epsilon). \tag{29}$$

Combining both equations, we obtain:

$$\frac{\sigma_t^2}{2\sigma_r^2} = \frac{1}{2}\frac{\int_0^\pi d\eta \sin^{\gamma+3}(\eta)}{\int_0^\pi d\eta \sin^{\gamma+1}(\eta)\cos^2(\eta)} = \frac{B\left(\frac{1}{2}, \frac{\gamma+4}{2}\right)}{B\left(\frac{3}{2}, \frac{\gamma+2}{2}\right)}, \tag{30}$$

where the function $B(z_1, z_2)$ is the beta function defined by:

$$B(z_1, z_2) = \int_0^1 t^{z_1-1}(1-t)^{z_2-1}\, dt \tag{31}$$

and is related to the Γ function by:

$$B(z_1, z_2) = \frac{\Gamma(z_1)\,\Gamma(z_2)}{\Gamma(z_1+z_2)}. \tag{32}$$

So we can write:

$$\frac{\sigma_t^2}{2\sigma_r^2} = \frac{1}{2}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{\gamma+4}{2})}{\Gamma(\frac{1}{2}+\frac{\gamma+4}{2})}\frac{\Gamma(\frac{3}{2}+\frac{\gamma+2}{2})}{\Gamma(\frac{3}{2})\Gamma(\frac{\gamma+2}{2})}. \tag{33}$$

Using:

$$\Gamma(z+1) = z\Gamma(z), \tag{34}$$

this simplifies to:

$$\frac{\sigma_t^2}{2\sigma_r^2} = \frac{\gamma+2}{2}. \tag{35}$$

And finally, we get:

$$\beta = 1 - \frac{\sigma_t^2}{2\sigma_r^2} = 1 - \frac{\gamma+2}{2} = -\frac{\gamma}{2}, \tag{36}$$

which is the expected result.

<u>Problem 4</u>:

The Jeans theorem states that any function of integrals of motion yields a steady-state solution of the collisonless Boltzmann equation. The question is thus is the distribution function (DF) a function of integrals of motion ? The potential being symmetric and time independent, the system yields at least four integrals of motion: the energy and the three components of the angular momentum.

Does the DF depend on the energy ? The answer is yes. As initially each collisionless test particles forming the sea arrive with velocities $\mathbf{v} = (v, 0, 0)$, the phase space is restricted to a sub-space of energy:

$$E = \frac{1}{2}V^2 = E_0. \tag{37}$$

Thus, the DF must have a dependency on the energy of the form:

$$f(\mathbf{x}, \mathbf{v}) \sim \delta(E - E_0). \tag{38}$$

However, it cannot be the only dependency. If it was, by symmetry, the mean velocity derived from the DF should be zero, as imposed by the symmetry of the ergodic DF. This is in contradiction with the fact that, by construction, the test particles have a no-zero mean velocity.

Does the DF depends on some components of the angular momentum ? If we assume the system to be invariant with respect to any rotation around the x axis, i.e., the dependency on the angular momentum can only be through L_x and $L_y^2+L_z^2$. As initially each collisionless test particles forming the sea arrive with velocities $\mathbf{v} = (v, 0, 0)$, the x-component of their angular momentum is necessarily zero and the DF must be of the form:

$$f(\mathbf{x}, \mathbf{v}) \sim \delta(E - E_0)\delta(L_x) = \delta(E - E_0)\delta(yv_z - zv_y). \tag{39}$$

Because the DF will be even in V_x, the mean x-velocity will be 0, which is in contradiction with the test particles having a no-zero mean velocity along the x-axis.

What about the dependency on $L_y^2+L_z^2$? In a cylindrical coordinate system aligned with the x axis, $L_y^2 + L_z^2$ writes:

$$L_y^2 + L_z^2 = v_x^2 r^2 + x^2 v_r^2 + x^2 v_\theta^2 - x v_x r v_r, \tag{40}$$

where r is the distance perpendicular to the x axis. The DF could thus write:

$$f(\mathbf{x}, \mathbf{v}) = \delta(E - E_0)\delta(r v_\theta) f_1(v_x^2 r^2 + x^2 v_r^2 + x^2 v_\theta^2 - x v_x r v_r). \tag{41}$$

Anywhere along the x axis ($r = 0$), the mean velocity along this axis is still 0, while we would expect a non-zero value. The reason for zero mean velocities along the x axis is that that the DF does not prevent particles to move with an opposite velocity. To remove those particles the DF should depends on:

$$f(\mathbf{x}, \mathbf{v}) \sim \delta(v_x - v)\delta(v_y)\delta(v_z), \tag{42}$$

but none of the velocity component are integrals of motion. Consequently, the distribution function of the test particles does not satisfy the Jeans theorem.