Equilibria and stability of collisionless systems

4st and 1st part

Outlines

The Virial Equation and Virial Theorem

- Theory
- Applications

N-body- experiments

- Are systems defined from a DF that solve the CB stable ?
- Comments and discussions on the experiments

Equilibria of collisionless systems

The Virial Theorem

Remainder : moments of the CB Equation

First moment

$$\frac{\partial}{\partial t} f + \sum_i v_i \frac{\partial f}{\partial x_i} - \sum_i \frac{\partial \phi}{\partial x_i}\frac{\partial f}{\partial v_i} = 0$$

multiply by v_j and integrate over velocities

$$\frac{\partial}{\partial t}\underbrace{\int d^3v \, v_j \, f}_{\nu \bar{v}_j} + \underbrace{\int d^3v \sum_i v_i v_j \frac{\partial f}{\partial x_i}}_{①} - \sum_i \frac{\partial \phi}{\partial x_i} \underbrace{\int d^3v \, v_j \frac{\partial f}{\partial v_i}}_{②} = 0$$

$$② = \frac{\partial \phi}{\partial x_j}\nu$$

① $$\int d^3v \sum_i v_i v_j \frac{\partial f}{\partial x_i} = \sum_i \frac{\partial}{\partial x_i}\int d^3v \, v_i v_j f = \sum_i \frac{\partial}{\partial x_i}\left(\overline{v_i v_j}\,\nu\right)$$

② $$\int d^3v \frac{\partial}{\partial v_i}(v_j f) = \underbrace{\int d^3v \, v_j \frac{\partial f}{\partial v_i}}_{②} + \underbrace{\int d^3v \, f \underbrace{\frac{\partial v_j}{\partial v_i}}_{\delta_{ij}}}_{\delta_{ij}\nu}$$

$$\oint d^2\xi_i \, v_j f = 0$$

$$\frac{\partial}{\partial t}\left(\bar{v}_j \nu\right) + \sum_i \frac{\partial}{\partial x_i}\left(\overline{v_i v_j}\,\nu\right) + \nu \frac{\partial \phi}{\partial x_j} = 0$$

$$\nu \frac{\partial}{\partial t}(\bar{v}_j) + \nu \sum_i \bar{v}_i \frac{\partial}{\partial x_i} \bar{v}_j = - \sum_i \frac{\partial}{\partial x_i}(\sigma_{ij}^2 \nu) - \nu \frac{\partial \Phi}{\partial x_j}$$

Jeans 1919

Interpretation

Euler equation in hydrodynamics

Lagrangian form

$$\frac{d}{dt}\vec{v} = -\frac{\vec{\nabla} P}{\rho} - \vec{\nabla}\phi$$

Eulerian form

(*) $$\frac{\partial}{\partial t}\vec{v} + \vec{v}\cdot\vec{\nabla}\vec{v} = -\frac{\vec{\nabla} P}{\rho} - \vec{\nabla}\phi$$

$$\rho \frac{\partial}{\partial t}\vec{v} + \rho\,\vec{v}\cdot\vec{\nabla}\vec{v} = -\vec{\nabla}P - \rho\,\vec{\nabla}\cdot\phi$$

"j" component only

$$\rho \frac{\partial}{\partial t} v_j + \rho \sum_i v_i \frac{\partial v_j}{\partial x_i} = -\frac{\partial P}{\partial x_j} - \rho \frac{\partial \Phi}{\partial x_j}$$

(*) $$\frac{dv_i(x,y,z)}{dt} = \frac{\partial v_i}{\partial t} + \sum \frac{\partial v_i}{\partial x}\dot{x}$$

Virial Equation - Virial Theorem (many different derivations exist)

- Integrate the moments of the CBM over the configuration and velocity space

$$\frac{\partial}{\partial t} f + \sum_i v_i \frac{\partial f}{\partial x_i} - \sum_i \frac{\partial \phi}{\partial x_i} \frac{\partial f}{\partial v_i} = 0$$

zeroth moment

$$\int d^3x \int d^3v \, \frac{\partial}{\partial t} f + \int d^3x \int d^3v \sum_i v_i \frac{\partial f}{\partial x_i} - \int d^3x \int d^3v \sum_i \frac{\partial \phi}{\partial x_i} \frac{\partial f}{\partial v_i} = 0$$

$\equiv$ integrate over $\vec{x}$ the 1st Jeans Eqn.

$$\frac{\partial}{\partial t} \rho + \vec{\nabla} \cdot (\rho \vec{\bar{v}}) = 0$$

$$\int d^3x \, \frac{\partial}{\partial t} \rho + \underbrace{\int d^3x \, \vec{\nabla} \cdot (\rho \vec{\bar{v}})}_{=0} = 0$$

$$\frac{dM}{dt} = 0$$

Total mass conservation

$\partial \to d$ as M no longer dep. on $\vec{x}, \vec{v}$

First moment multiply by $x_k v_j$ and integrate

$$\int d^3x \int d^3v\, x_k v_j \frac{\partial}{\partial t} f + \int d^3x \int d^3v\, x_k v_j \sum_i v_i \frac{\partial f}{\partial x_i} - \int d^3x \int d^3v\, x_k v_j \sum_i \frac{\partial \phi}{\partial x_i} \frac{\partial f}{\partial v_i} = 0$$

$\equiv$ multiply by x_k and integrate over $\vec{x}$ the 2nd Jeans Eqn. (momentum conservation)

$$\frac{\partial}{\partial t}(\bar{v}_j \rho) + \sum_i \frac{\partial}{\partial x_i}\left(\overline{v_i v_j}\, \rho\right) + \rho \frac{\partial \phi}{\partial x_j} = 0$$

$\left[\frac{g}{cm^3} \cdot \frac{N}{g}\right] \equiv \left[\frac{g}{cm^2\, s^2}\right]$

$\Rightarrow$ Energy equation ($\frac{\partial}{\partial t} \vec{p} \cdot \vec{x} \equiv$ energy)

$$\int d^3\vec{x}\, x_k \frac{\partial}{\partial t}(\bar{v}_j \rho) = -\int d^3\vec{x}\, x_k \sum_i \frac{\partial}{\partial x_i}\left(\overline{v_i v_j}\, \rho\right) - \int d^3\vec{x}\, x_k\, \rho \frac{\partial \phi}{\partial x_j}$$

$\left[g \frac{cm^2}{s^2}\right]$

$$\frac{d}{dt}\int d^3\vec{x}\, x_k (\bar{v}_j \rho)$$

$\frac{d}{dt} x_i = 0$

(*)

div. th. $\int d^3\vec{x}\, g\, \vec{\nabla}\cdot\vec{F} = \oint g\, \vec{F}\cdot d\vec{S} - \int d^3\vec{x}\; \vec{F}\cdot\vec{\nabla} g$

$$(*)\quad \int d^3\vec{x}\, x_k \underbrace{\sum_i \frac{\partial}{\partial x_i}\left(\overline{v_i v_j}\,\rho\right)}_{\vec{\nabla}\cdot\left(\overline{v_i v_j}\,\rho\right)} \overset{!}{=} \underbrace{\oint d^2\vec{S}\, x_k\, \overline{v_i v_j}\,\rho}_{=0 \quad v=0 \quad x\to\infty} - \int d^3\vec{x} \sum_i \rho\, \overline{v_i v_j}\, \overbrace{\frac{\partial x_k}{\partial x_i}}^{\delta_{ki}}$$

$$= -\int d^3\vec{x}\; \overline{v_k v_j}\;\rho$$

Equation for the energy

$$\frac{d}{dt}\int d^3\vec{x}\; x_k\left(\bar{v}_j\,\rho\right) = \int d^3\vec{x}\;\overline{v_k v_j}\;\rho - \int d^3\vec{x}\; x_k\,\rho\,\frac{\partial\phi}{\partial x_j}$$

$\left[g\,\frac{cm^2}{s^2} \right]$

Definitions

symmetric tensor

① Kinetic energy tensor

$$K_{jk} := \frac{1}{2}\iint d^3\vec{x}\, d^3\vec{v}\, v_i v_j\, f = \frac{1}{2}\int d^3\vec{x}\; \overline{v_j v_k}\; \rho$$

with $\sigma_{jk}^2 = \frac{1}{\rho(\vec{x})}\int d^3\vec{v}\; (v_j - \bar{v}_j)(v_k - \bar{v}_k)\; f(\vec{x},\vec{v})$

$$= \overline{v_j v_k} - \bar{v}_j\, \bar{v}_k$$

$$K_{jk} = \underbrace{\frac{1}{2}\int d^3\vec{x}\; \rho\; \bar{v}_j \bar{v}_k}_{T_{jk}} + \underbrace{\frac{1}{2}\int d^3\vec{x}\; \rho\, \sigma_{jk}^2}_{\Pi_{jk}}$$

$$K_{jk} = T_{jk} + \frac{1}{2}\Pi_{jk}$$

ordered motions / random motions

Trace $\mathrm{Tr}(K_{jk}) := K = \sum_j \frac{1}{2}\int d^3\vec{x}\; \overline{v_j^2}\; \rho$: Total kinetic energy

Definitions

symmetric tensor

② Potential energy tensor

$$W_{jk} := -\iint d^3\vec{x}\, d^3\vec{v}\, \rho\, x_k \frac{\partial \phi}{\partial x_j} = -\int d^3\vec{x}\, \rho\, x_k \frac{\partial \phi}{\partial x_j}$$

With $\phi(\vec{x}) = -G \int d^3\vec{x}' \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|}$

$$W_{jk} = G \int d^3\vec{x}\, \rho(\vec{x})\, x_k \frac{\partial}{\partial x_j} \int d^3\vec{x}' \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|}$$

move integral + derivate

a) $$= G \iint d^3\vec{x}\, d^3\vec{x}'\, \rho(\vec{x})\, \rho(\vec{x}') \frac{x_k (x_j' - x_j)}{|\vec{x}' - \vec{x}|^3}$$

change variables name $\vec{x}' \leftrightarrow \vec{x}$

b) $$= G \iint d^3\vec{x}\, d^3\vec{x}'\, \rho(\vec{x}')\, \rho(\vec{x}) \frac{x_k' (x_j - x_j')}{|\vec{x} - \vec{x}'|^3}$$

Summing $\frac{1}{2}$ a) + $\frac{1}{2}$ b) and $x_k(x_j'-x_j) + x_k'(x_j-x_j') \equiv -(x_k'-x_k)(x_j'-x_j)$

$$W_{jk} = -\frac{1}{2} G \int d^3\vec{x} \int d^3\vec{x}'\, \rho(\vec{x})\rho(\vec{x}') \frac{(x_j'-x_j)(x_k'-x_k)}{|\vec{x}-\vec{x}'|^3}$$

Trace $\quad Tr(W_{jk}) := W \qquad \sum_j (x_j'-x_j)^2 = (\vec{x}'-\vec{x})^2$

$$= -\frac{1}{2} G \int d^3\vec{x} \int d^3\vec{x}'\, \rho(\vec{x})\rho(\vec{x}') \frac{1}{|\vec{x}'-\vec{x}|}$$

$$= \frac{1}{2} \int d^3\vec{x}\, \rho(\vec{x}) \underbrace{\left[-G \int d^3\vec{x}'\, \rho(\vec{x}') \frac{1}{|\vec{x}'-\vec{x}|} \right]}_{=\phi(\vec{x})}$$

$$W = \frac{1}{2} \int d^3\vec{x}\, \rho(\vec{x})\, \phi(\vec{x}) \equiv -\int d^3\vec{x}\, \rho\, \vec{x}\cdot\vec{\nabla}\phi(\vec{x})$$

From the definition of W_{jk}

Total gravitational potential

Definitions

symmetric tensor

(3) Inertial tensor

$$I_{jk} := \iint d^3\bar{x}\, d^3\bar{v}\, f(\bar{x})\, x_j x_k = \int d^3\bar{x}\, \rho(\bar{x})\, x_j x_k$$

Time derivative

$$\frac{d}{dt} I_{jk} = \int d^3\bar{x}\, x_j x_k \frac{\partial}{\partial t} \rho(\bar{x})$$

$\frac{d}{dt} x_i = 0$

continuity equation
"zeroth moment of the CB"

$$\frac{\partial}{\partial t} \rho(\bar{x}) + \vec{\nabla}\cdot(\rho \bar{v}) = 0$$

$$\frac{d}{dt} I_{jk} = -\int d^3\bar{x}\, x_j x_k\, \vec{\nabla}\cdot(\rho\bar{v})$$

$$= \sum_i \int d^3\bar{x}\, \rho \bar{v}_i \left(x_k \delta_{ji} + x_j \delta_{ki} \right)$$

divergeance theorem

$\int d^3x\, g \vec{\nabla}\vec{F} = \oint d^2S\, g \vec{F} - \int d^3x\, \vec{F}\cdot\vec{\nabla} g$

$$\boxed{\frac{d}{dt} I_{jk} = \int d^3\bar{x}\, \rho \left(\bar{v}_j x_k + \bar{v}_k x_j \right)}$$

With those definitions and results, the "energy equation" becomes

$$\frac{d}{dt}\int d^3\vec{x}\; x_k\left(\bar{v}_j \rho\right) = \int d^3\vec{x}\; \overline{v_k v_j}\,\rho - \int d^3\vec{x}\; x_k\,\rho\,\frac{\partial\phi}{\partial x_j}$$

$$\frac{d}{dt}\int d^3\vec{x}\; x_k\left(\bar{v}_j \rho\right) = 2K_{kj} + W_{kj}$$

$$= 2T_{kj} + \Pi_{kj} + W_{kj}$$

Now, we "average" the (k,j) and (j,k) components: $\frac{1}{2}Eq_{jk} + \frac{1}{2}Eq_{kj}$

$$\frac{1}{2}\frac{d}{dt}\underbrace{\int d^3\vec{x}\;\rho\left(x_k\bar{v}_j + x_j\bar{v}_k\right)} = \underbrace{T_{kj} + T_{jk}}_{\text{sym.}} + \underbrace{\frac{1}{2}\left(\Pi_{kj} + \Pi_{jk}\right)}_{\text{sym.}} + \underbrace{\frac{1}{2}\left(W_{kj} + W_{jk}\right)}_{\text{sym.}}$$

$$\frac{1}{2}\frac{d}{dt}\left(\frac{d}{dt} I_{jk}\right) = 2T_{jk} + \Pi_{jk} + W_{jk}$$

Virial "tensor" theorem

$$\frac{1}{2}\frac{d^2}{dt^2}\left(I_{jk} \right) = 2\,K_{jk} + W_{jk}$$

variation of the system shape	kinetic energy	potential energy

$$\frac{1}{2}\frac{d^2}{dt^2}\left(I_{jk} \right) = 2\,T_{jk} + \Pi_{jk} + W_{jk}$$

variation of the system shape	"ordered" kinetic energy	"random" kinetic energy	potential energy

If the system is at equilibrium

$$2\,K_{jk} + W_{jk} = 0 \quad \equiv \quad 2\,T_{jk} + \Pi_{jk} + W_{jk} = 0$$

Virial "scalar" theorem

Tr (Virial "tensor" theorem)

$$\frac{1}{2}\frac{d^2}{dt^2} I = 2K + W$$

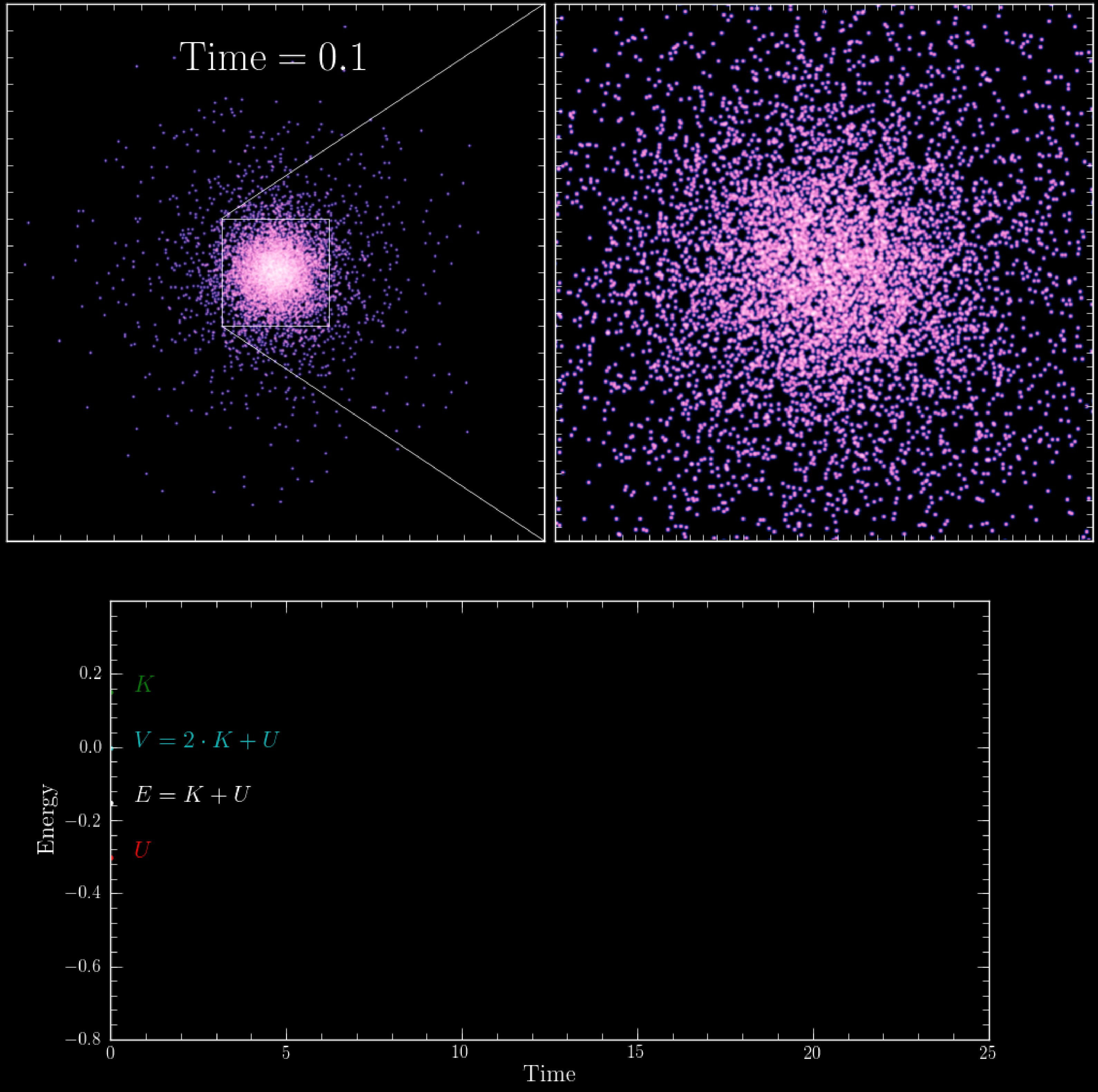
with $I = \sum_i I_{ii}$

If the system is at equilibrium

$(\ddot{I} = 0)$

$$2K + W = 0$$

total kinetic energy — total potential energy


Time = 0.1
K
$V = 2 \cdot K + U$
$E = K + U$
U
Energy
Time
0.2
0.0
−0.2
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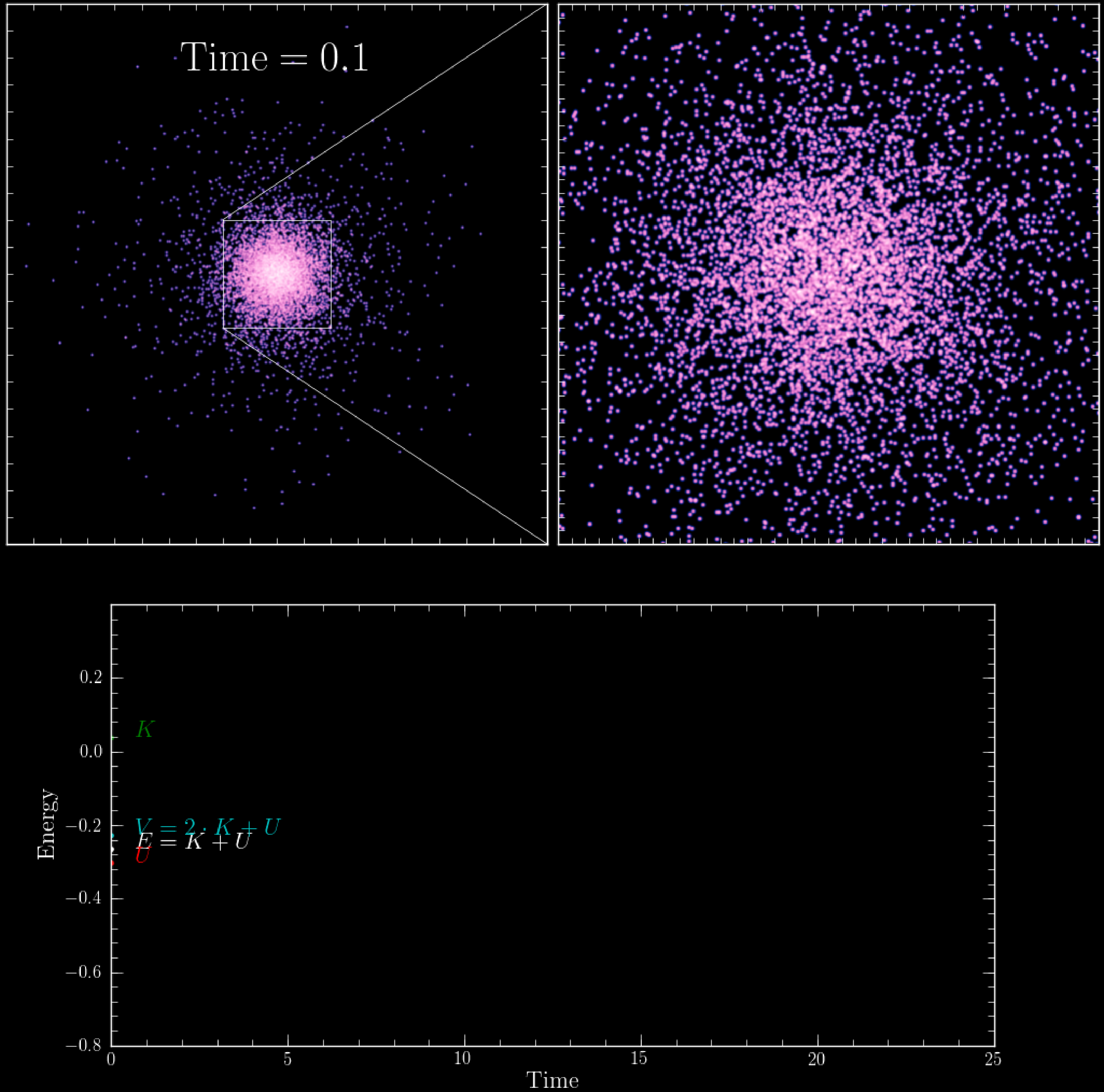
−0.6
−0.8
0
5
10
15
20
25


Time = 0.1
K
V = 2 · K + U
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Energy
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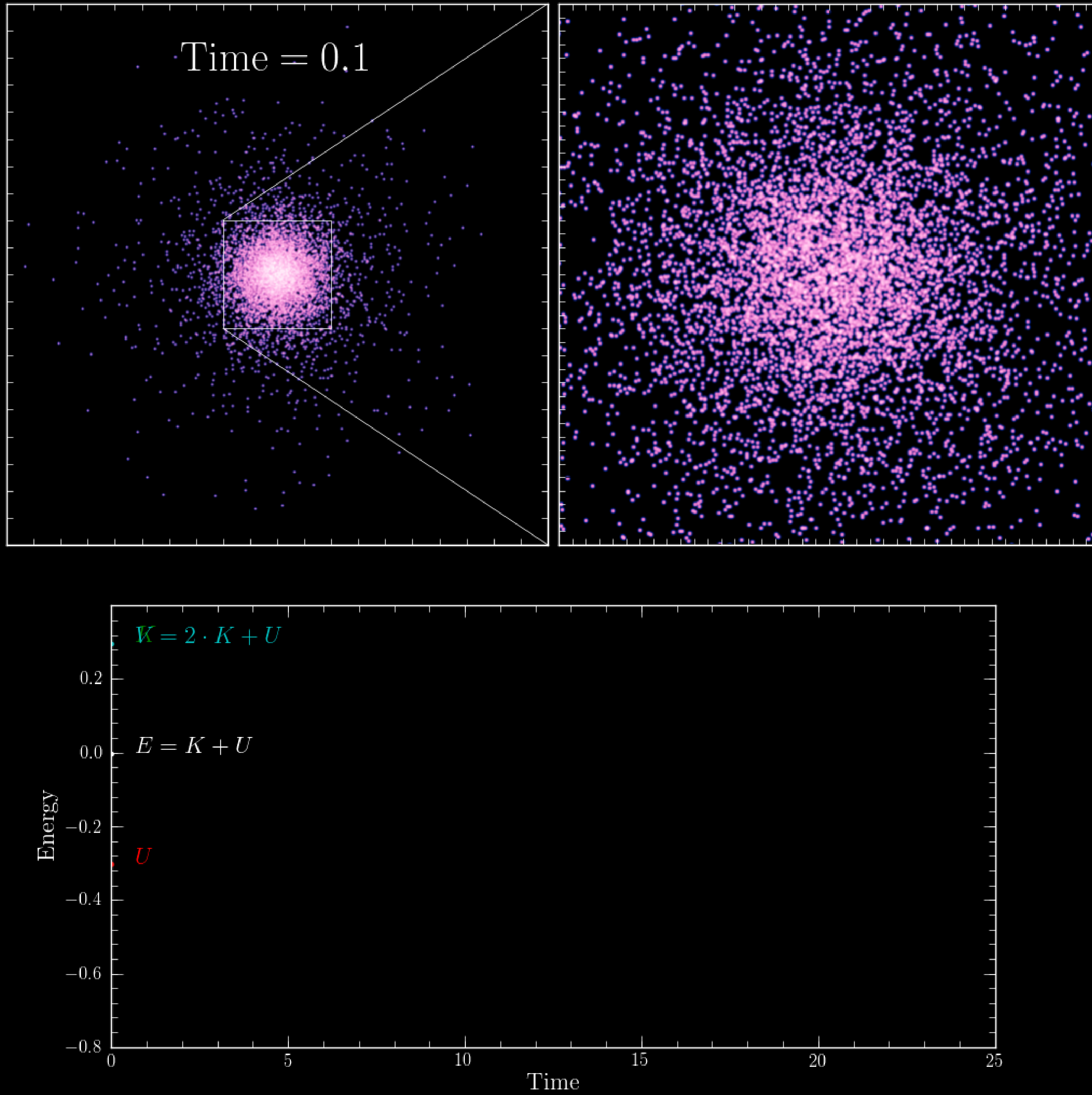
Time = 0.1
K
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U
Energy
0.2
0.0
−0.2
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−0.8
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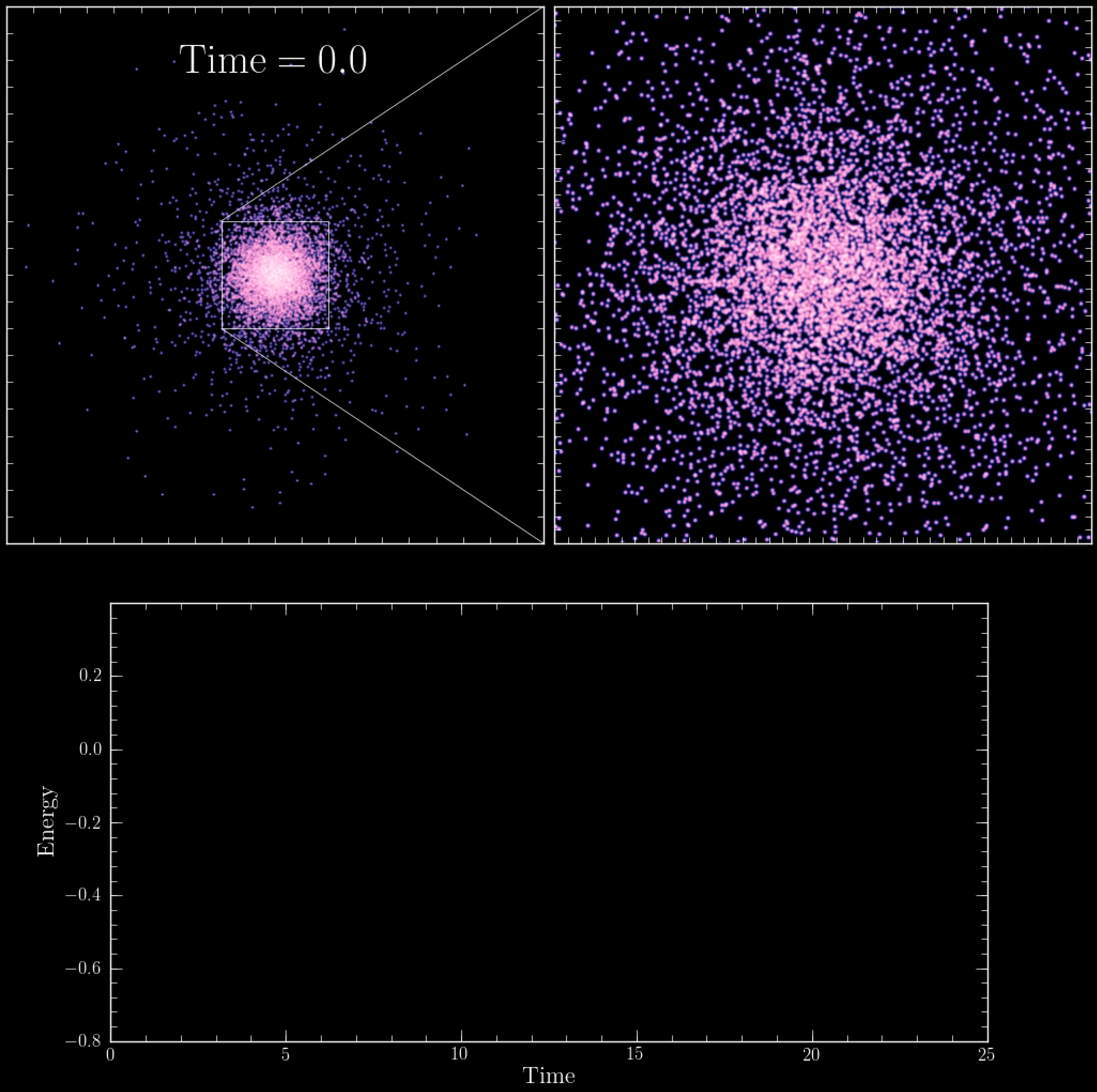
5
10
15
20
25
Time



Time = 0.0
Energy
0.2
0.0
−0.2
−0.4
−0.6
−0.8
0
5
10
15
20
25
Time

Isolated system of total energy E

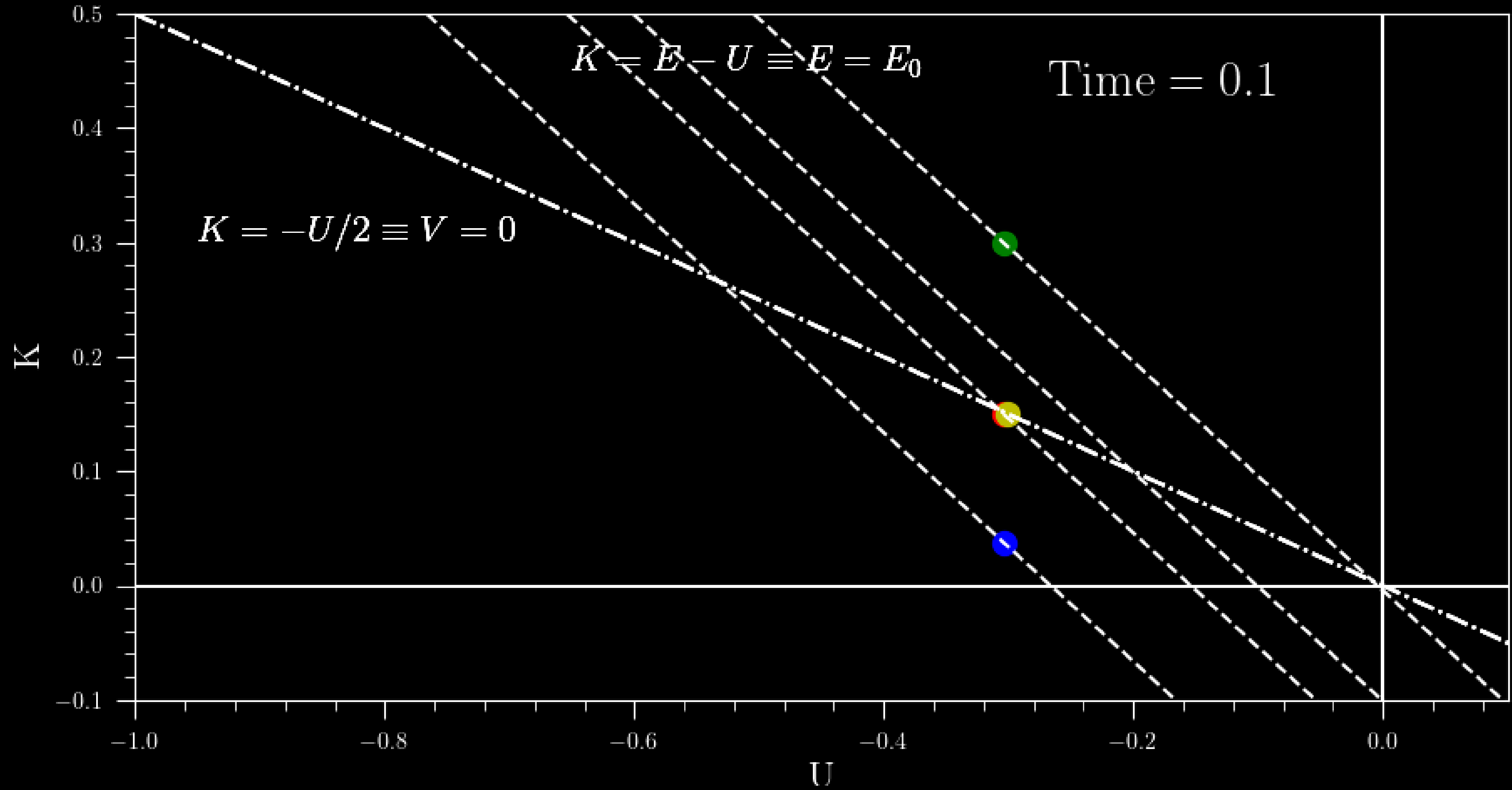
$$E = K + W$$

If the system is at the virial equilibrium ($\dot{I} = 0$)

$$2K + W = 0$$

Then

$$E = -K = \frac{1}{2} W$$



$K = E - U \equiv E = E_0$
Time = 0.1
$K = -U/2 \equiv V = 0$
K
U
0.5
0.4
0.3
0.2
0.1
0.0
−0.1
−1.0
−0.8
−0.6
−0.4
−0.2
0.0

Application

mass of the system

$$K \sim \frac{1}{2} M \langle v^2 \rangle$$

$$W \sim \frac{GM^2}{r_g}$$

r_g : the gravitational radius

$$r_g = \frac{GM^2}{|W|}$$

$$\boxed{\langle v^2 \rangle = \frac{|W|}{M} = \frac{GM}{r_g}}$$

If we measure $\langle v^2 \rangle$, we can access the system mass

- zeroth order : $r_g \sim$ size of the system
- first order : $r_g \sim \frac{1}{2} r_h$ (half mass radius)

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ON THE MASSES OF NEBULAE AND OF CLUSTERS OF NEBULAE

F. ZWICKY

ABSTRACT

Present estimates of the masses of nebulae are based on observations of the *luminosities* and *internal rotations* of nebulae. It is shown that both these methods are unreliable; that from the observed luminosities of extragalactic systems only lower limits for the values of their masses can be obtained (sec. i), and that from internal rotations alone no determination of the masses of nebulae is possible (sec. ii). The observed internal motions of nebulae can be understood on the basis of a simple mechanical model, some properties of which are discussed. The essential feature is a central core whose internal *viscosity* due to the gravitational interactions of its component masses is so high as to cause it to rotate like a solid body.

In sections iii, iv, and v three new methods for the determination of nebular masses are discussed, each of which makes use of a different fundamental principle of physics.

Method iii is based on the *virial theorem* of classical mechanics. The application of this theorem to the Coma cluster leads to a minimum value $\overline{M} = 4.5 \times 10^{10} M_{\odot}$ for the average mass of its member nebulae.

Fritz Zwicky (1898-1974): un personnage haut en couleurs. Prédit les étoiles à neutrons en tant que « cadavres » de supernovae (auxquelles il attribua aussi l'origine des rayons cosmiques), découvrit la matière noire des amas de galaxies et prédit les effets de mirage gravitationnel par les amas de galaxies.

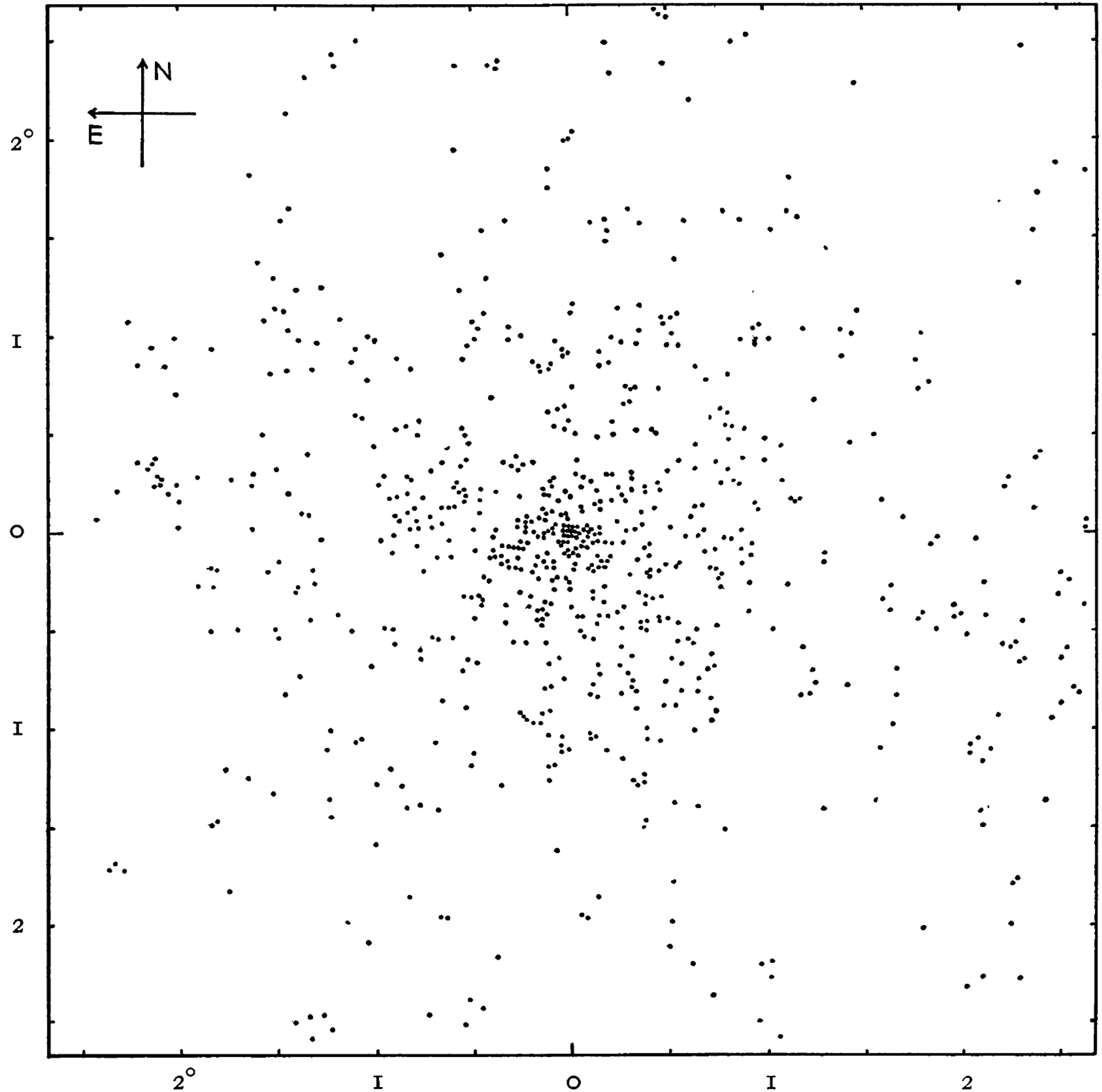


FIG. 3.—The Coma cluster of nebulae


The Coma cluster
Russ Carroll, Robert Gendler, and Bob Frank

We apply this relation to the Coma cluster of nebulae whose radius is of the order of 2×10^6 light-years. From the observational data we do not know directly the velocities v of the individual nebulae relative to the center of mass of the cluster. Only the velocity components v_s along the line of sight from the observer are known from the observed spectra of cluster nebulae. For a velocity distribution of spherical symmetry, however, we have

$$\overline{\overline{v^2}} = 3\overline{\overline{v_s^2}} . \tag{32}$$

Therefore

$$\mathcal{M} > \frac{3R\overline{\overline{v_s^2}}}{5\Gamma} . \tag{33}$$

From the observations of the Coma cluster so far available we have, approximately,[5]

$$\overline{\overline{v_s^2}} = 5 \times 10^{15} \text{cm}^2 \text{ sec}^{-2} . \qquad \text{\textasciitilde 700 km/s} \tag{34}$$

This average has been calculated as an average of the velocity squares alone without assigning to them any mass weights, as actually should be done according to (21). It seems, however, as Sinclair Smith[8] has shown for the Virgo cluster, that the velocity dispersion for bright nebulae is about the same as that for faint nebulae. Assuming this to be true also for the Coma cluster, it follows that the

[8] *Ap. J.*, **83**, 499, 1936.

Combining (33) and (34), we find

$$\mathcal{M} > 9 \times 10^{46} \text{gr} . \tag{35}$$

The Coma cluster contains about one thousand nebulae. The average mass of one of these nebulae is therefore

$$\overline{M} > 9 \times 10^{43} \text{ gr} = 4.5 \times 10^{10} M_{\odot} . \tag{36}$$

Inasmuch as we have introduced at every step of our argument inequalities which tend to depress the final value of the mass $\mathcal{M}$, the foregoing value (36) should be considered as the lowest estimate for the average mass of nebulae in the Coma cluster. This result is somewhat unexpected, in view of the fact that the luminosity of an average nebula is equal to that of about 8.5×10^7 suns. According to (36), the conversion factor γ from luminosity to mass for nebulae in the Coma cluster would be of the order

M/L

$$\gamma = 500 , \tag{37}$$

as compared with about $\gamma' = 3$ for the local Kapteyn stellar system.

The stability of collisionless systems

1st part

Stability of collisionless systems

Playing with N-body models

The Jeans equations for spherical systems

$$\frac{\partial}{\partial r}\left(\nu\sigma_r^2\right) + 2\frac{\beta}{r}\nu\sigma_r^2 = -\nu\frac{\partial\Phi}{\partial r}$$

$$r^{-2\beta}\frac{\partial}{\partial r}\left(\nu\sigma_r^2 r^{2\beta}\right) = -\nu\frac{\partial\Phi}{\partial r}$$

If the system has a constant anisotrpy parameter $\beta = cte$ $\beta := 1 - \frac{\sigma_\theta^2 + \sigma_\phi^2}{2\sigma_r^2} = 1 - \frac{\sigma_t^2}{2\sigma_r^2}$

$$\sigma_r^2(r) = \frac{1}{r^{2\beta}\nu(r)}\int_r^\infty \mathrm{d}r' r'^{2\beta}\nu(r')\frac{\partial\Phi}{\partial r'} = \frac{G}{r^{2\beta}\nu(r)}\int_r^\infty \mathrm{d}r' r'^{2\beta-2}\nu(r')M(r')$$

$\beta = -\infty$

- Circular orbits
  $\sigma_\theta = \sigma_\phi \neq 0, \sigma_r = 0$

$\beta = 0$

- Isotrope ergodic
  $\sigma_\theta = \sigma_\phi = \sigma_r = \frac{1}{\sqrt{2}}\sigma_t$

$\beta = 1$

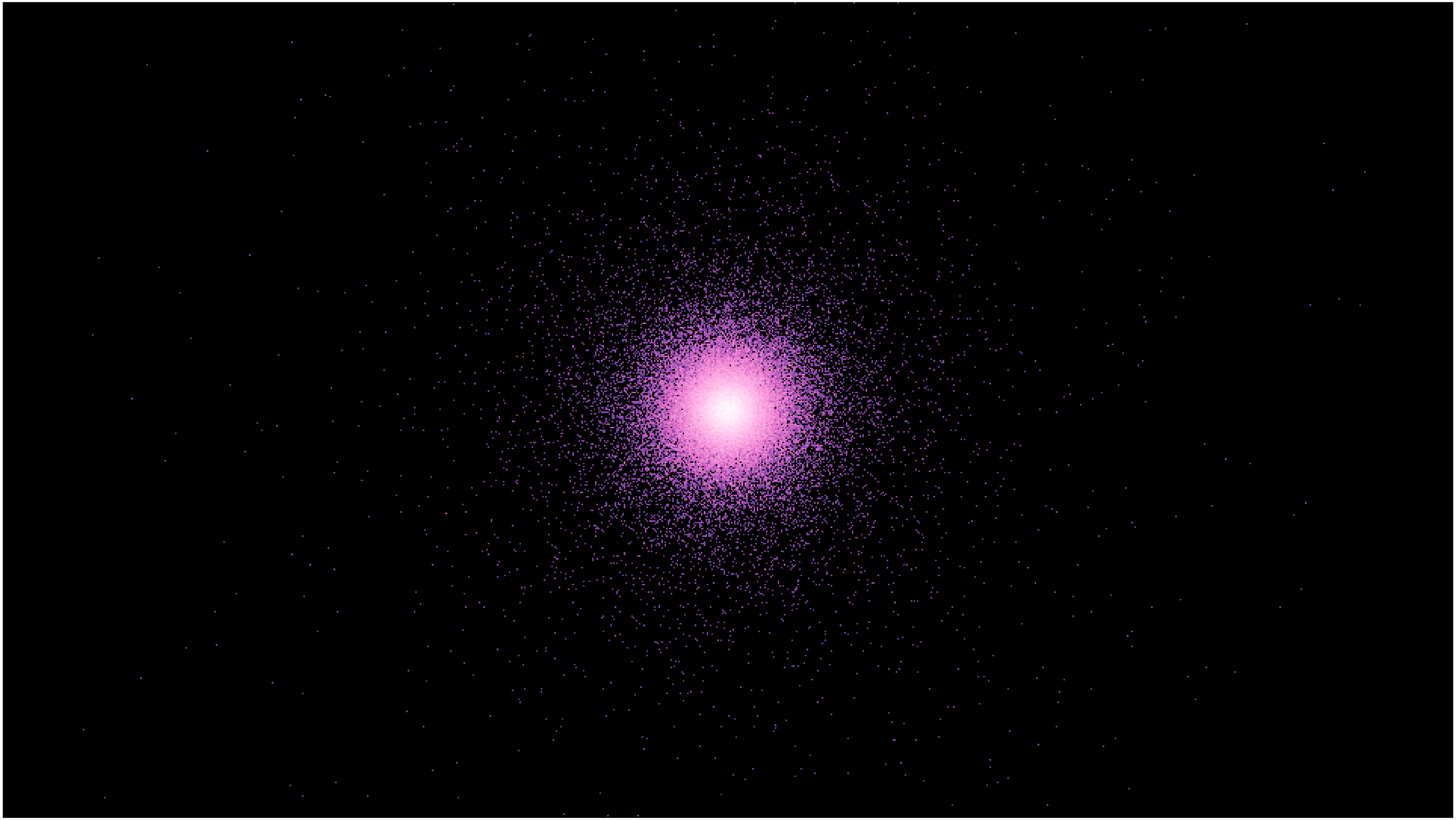
- Radial orbits
  $\sigma_\theta = \sigma_\phi = 0, \sigma_r \neq 0$

- tangentially biased orbits
  $\sigma_\theta = \sigma_\phi > \sigma_r$
- radially biased orbits
  $\sigma_\theta = \sigma_\phi < \sigma_r$

Plummer model

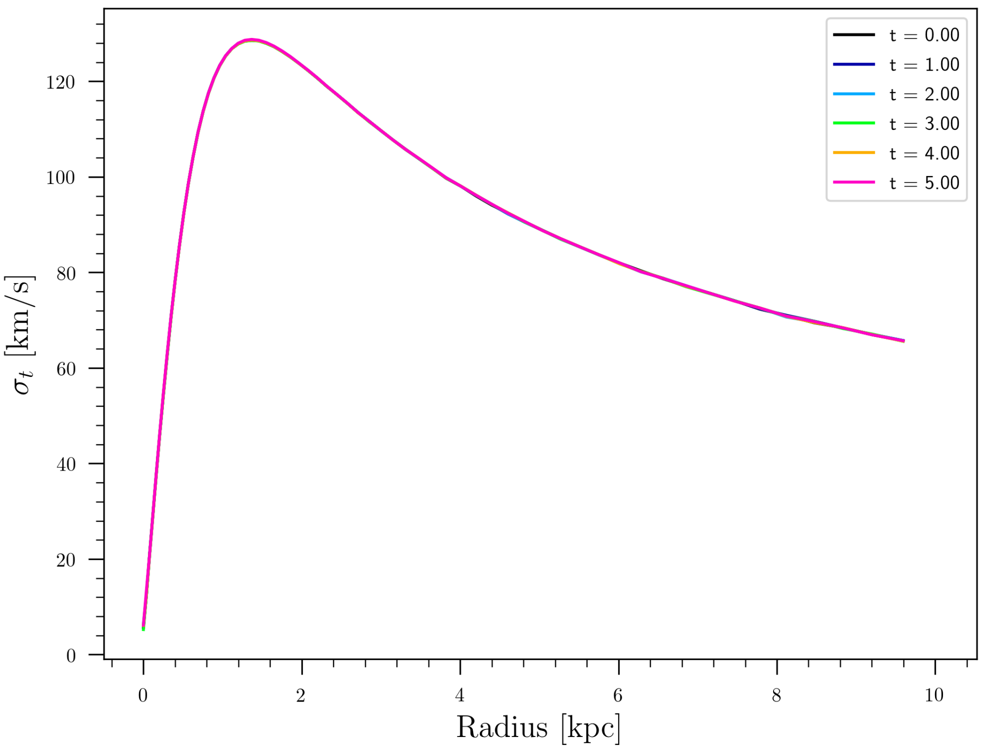
$$\beta = -\infty \;\Rightarrow\; \sigma_t \neq 0, \sigma_r = 0$$

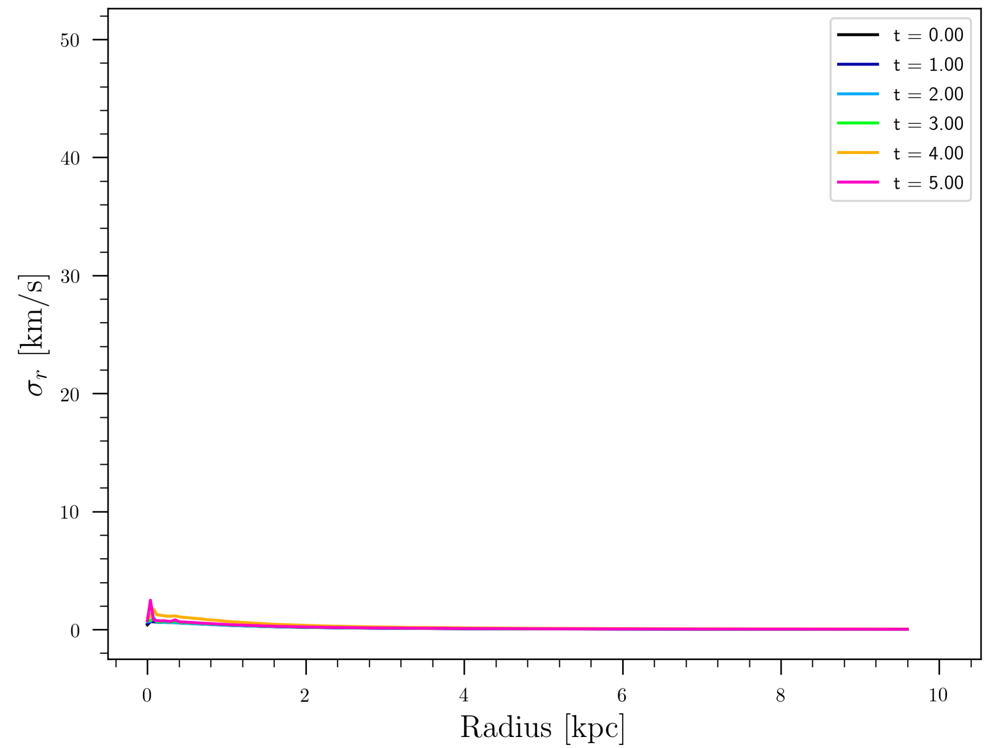
Not self-gravitating !



Plummer model

$$\beta = -\infty \;\Rightarrow\; \sigma_t \neq 0, \sigma_r = 0$$

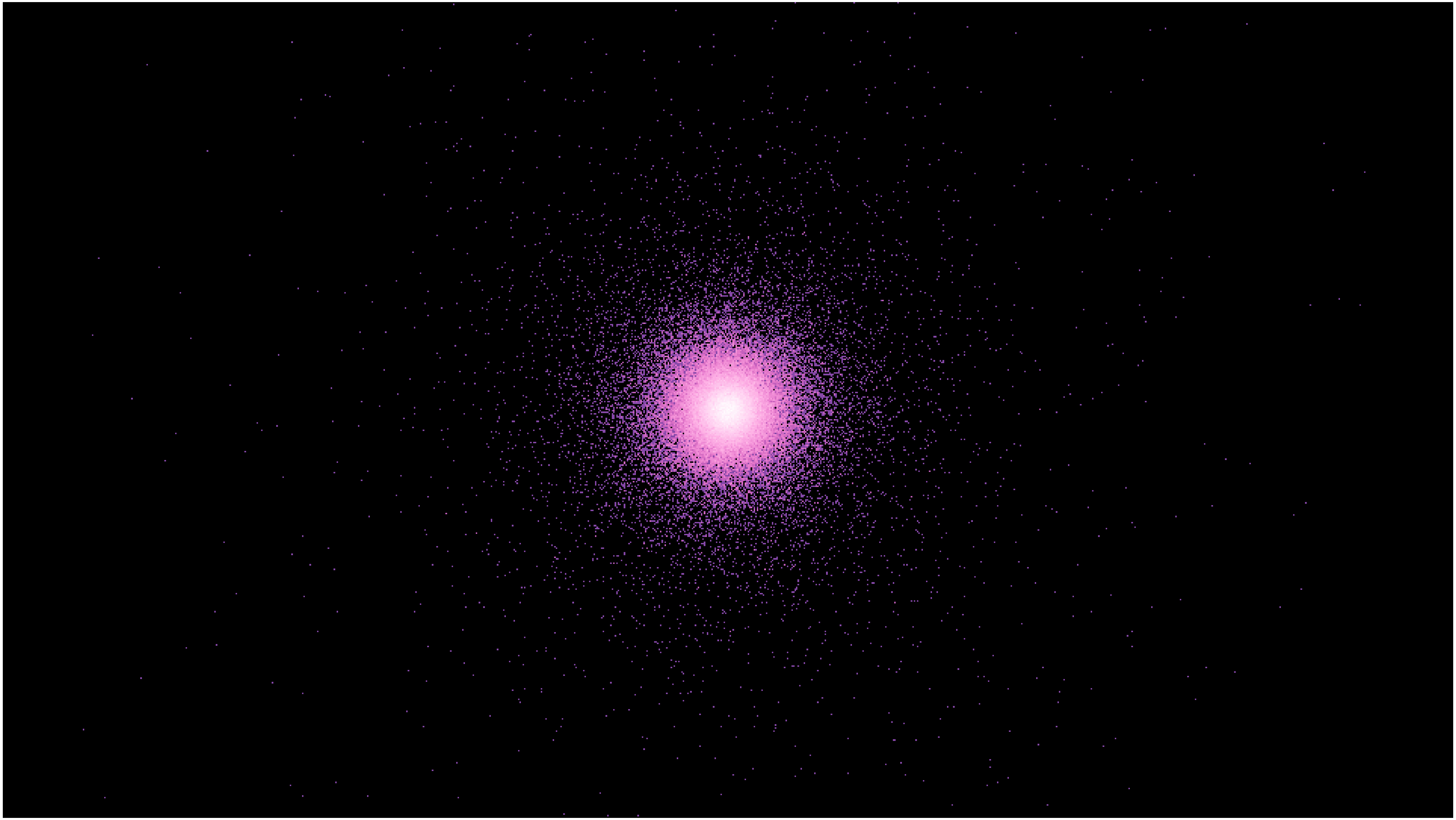




Plummer model

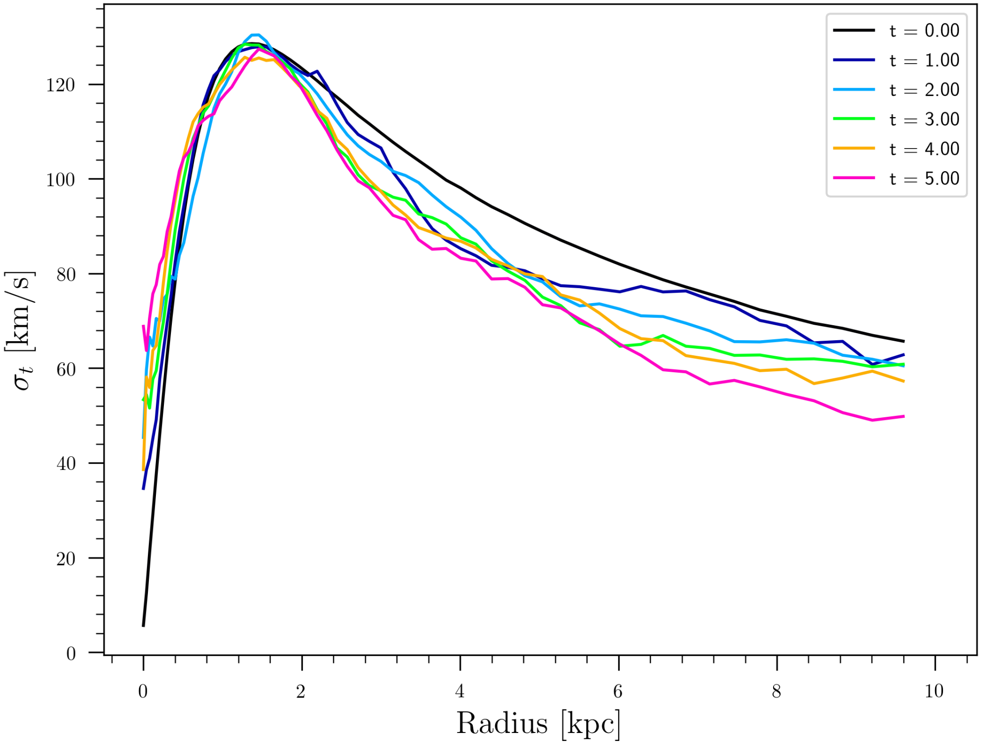
$$\beta = -\infty \Rightarrow \sigma_t \neq 0, \sigma_r = 0$$

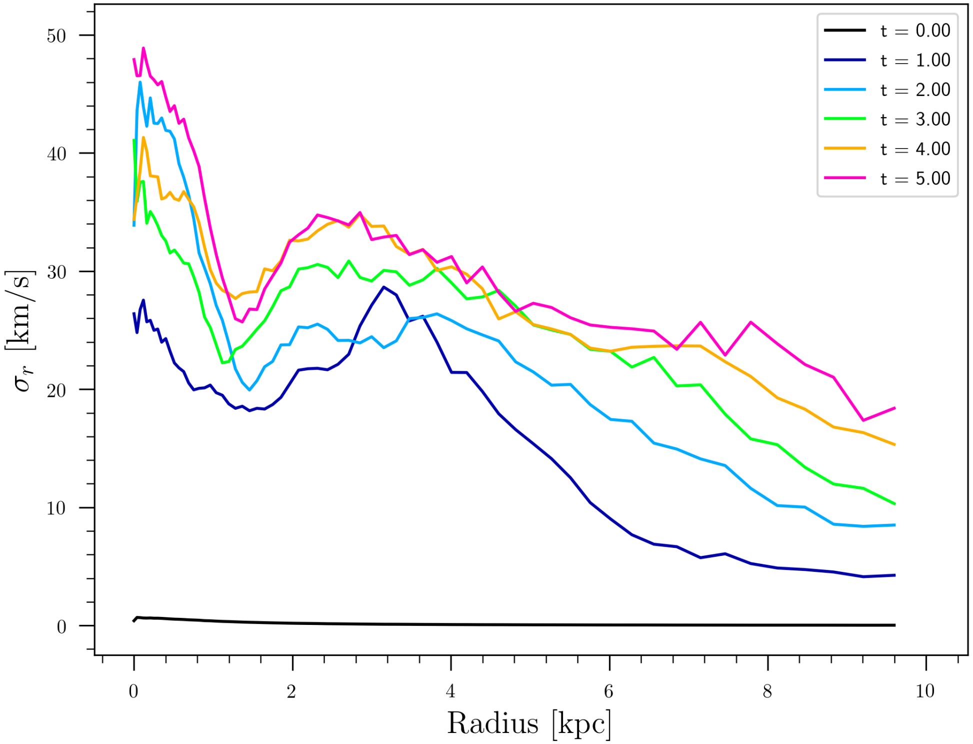
Self-gravitating !



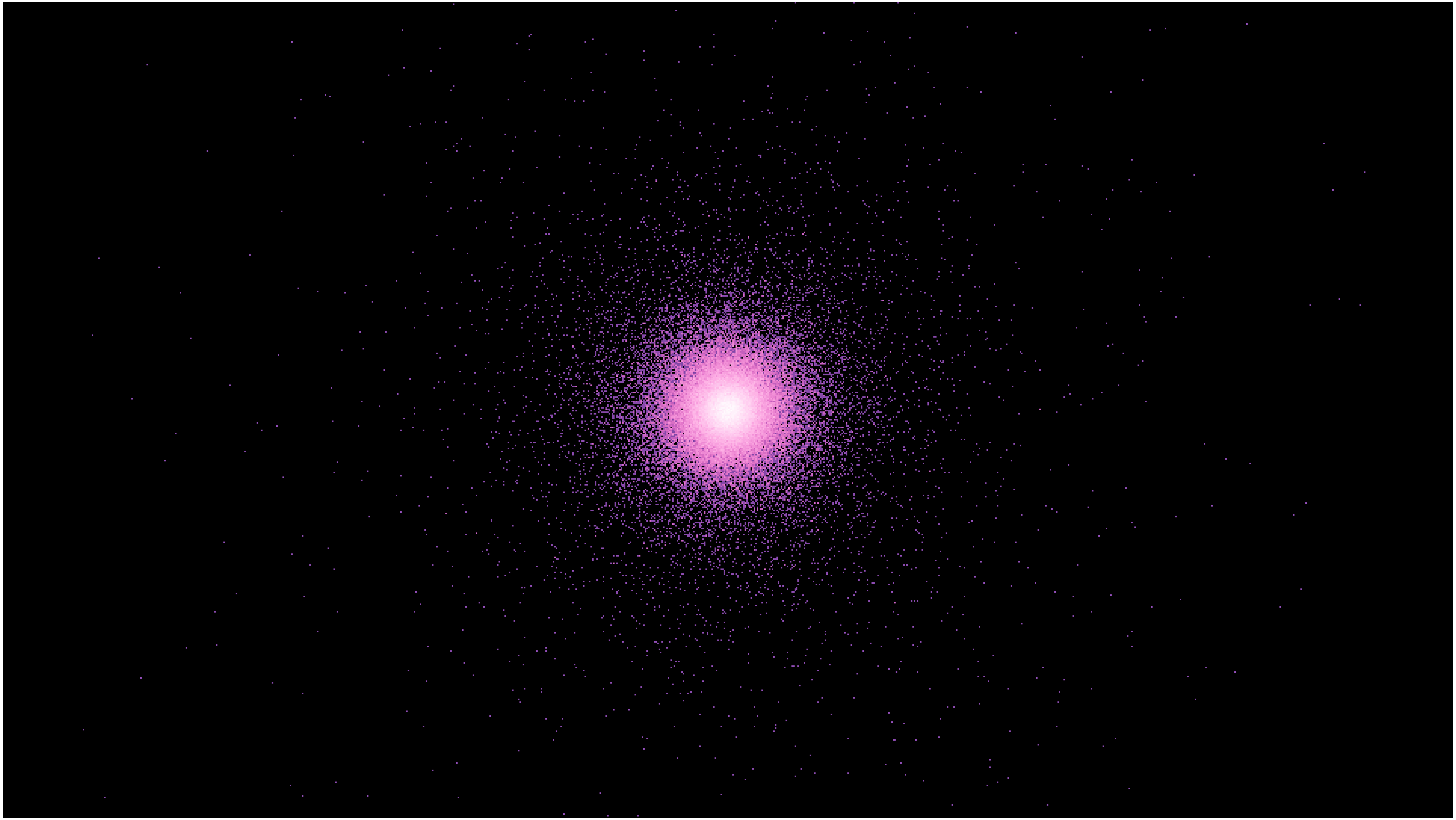
Plummer model

$$\beta = -\infty \Rightarrow \sigma_t \neq 0, \sigma_r = 0$$

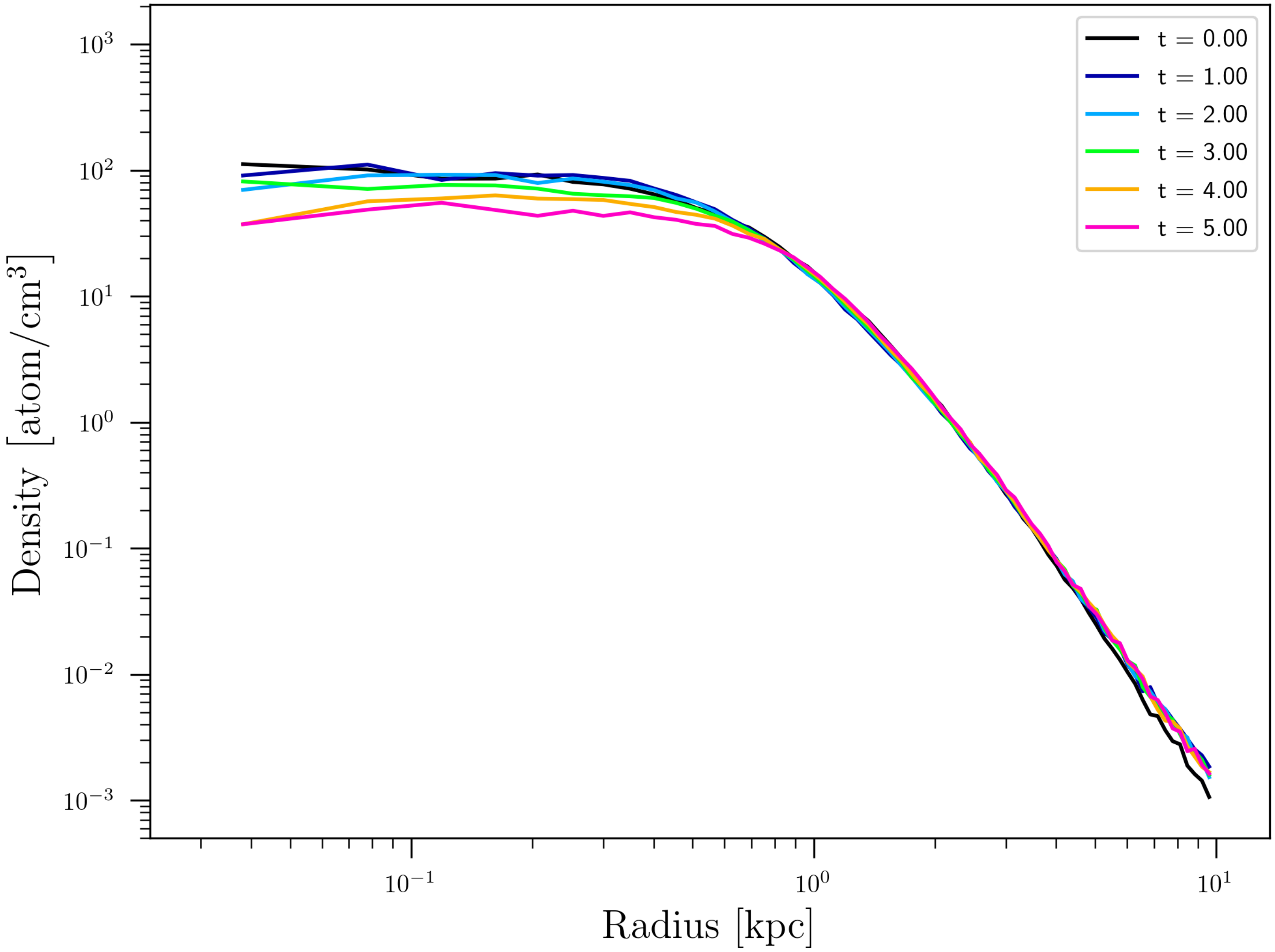




Plummer model with intermediate anisotropy $\beta = -1$



Plummer model with intermediate anisotropy $\beta = -1$



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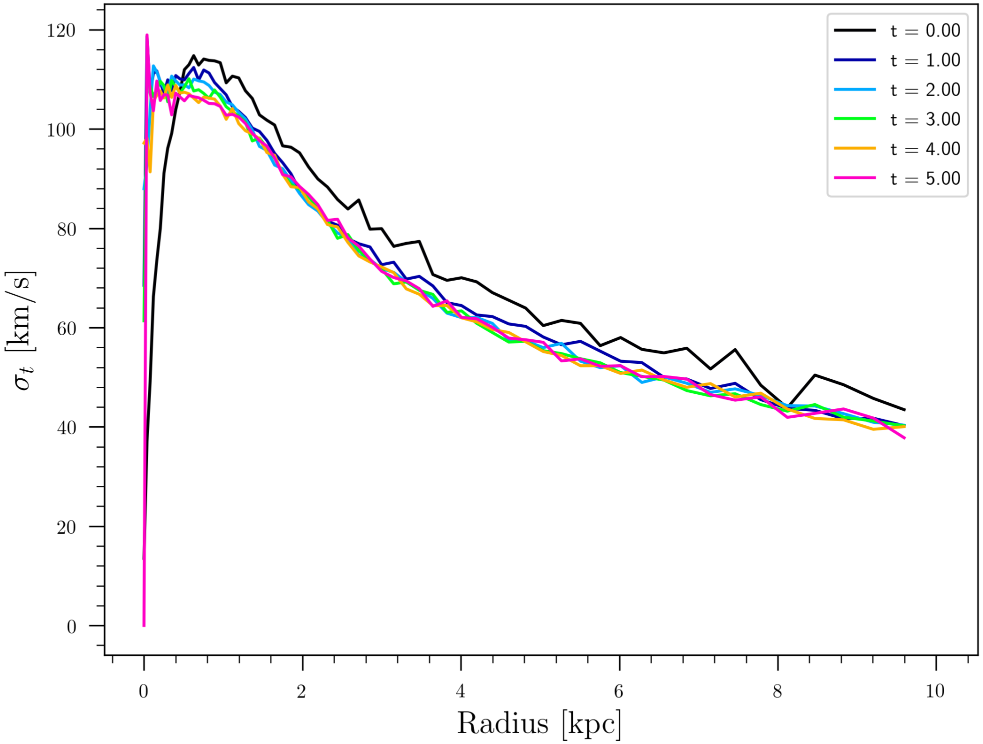

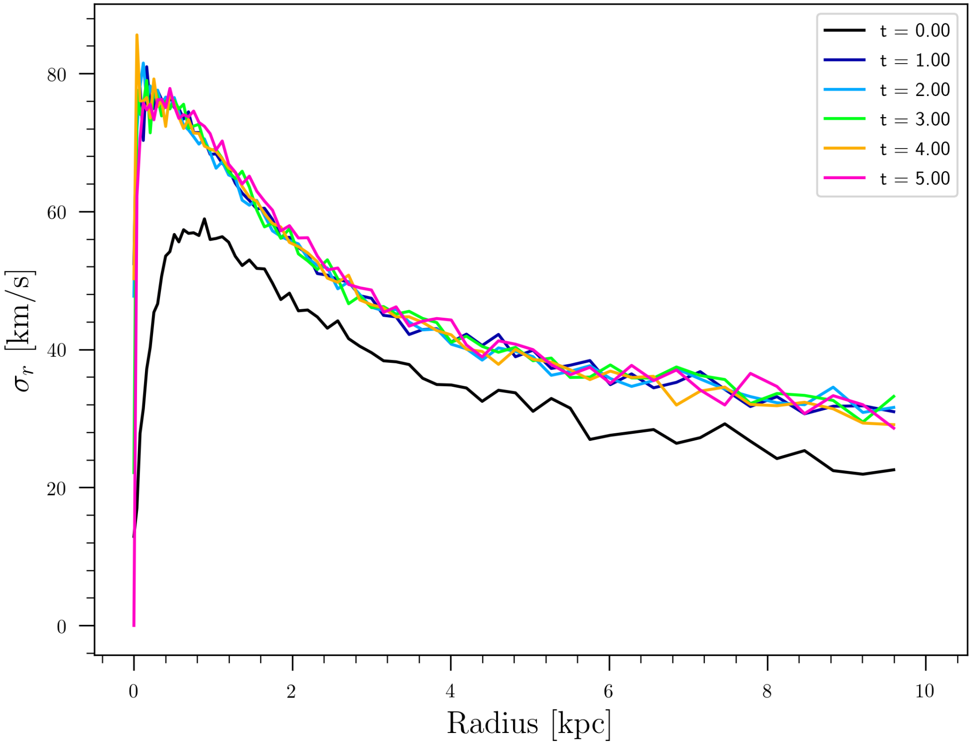

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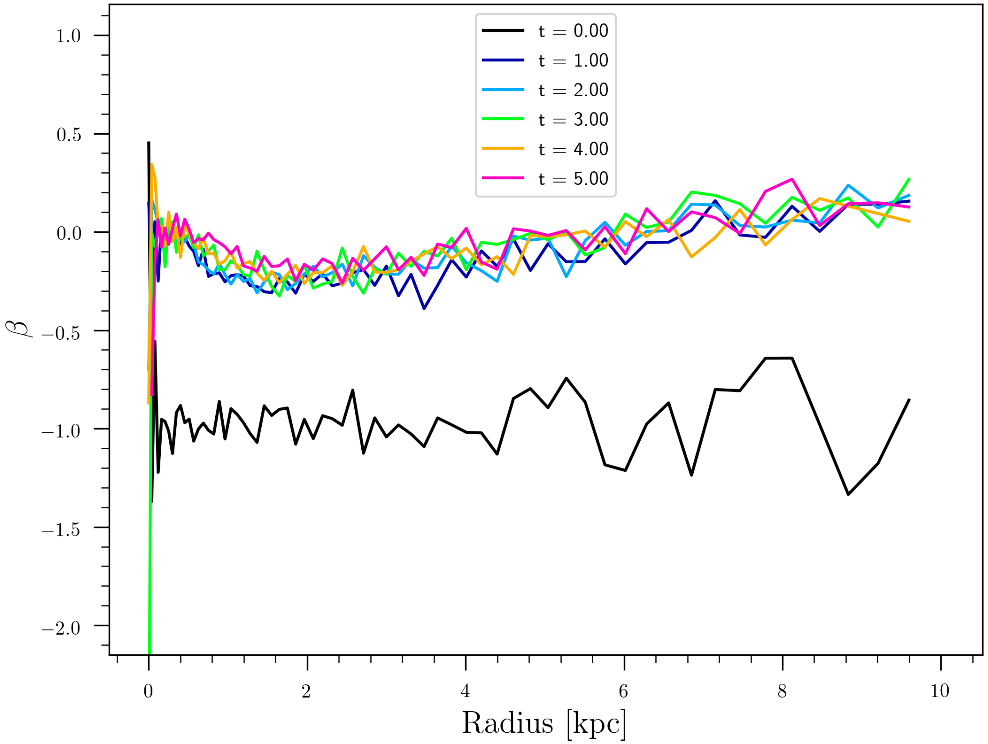
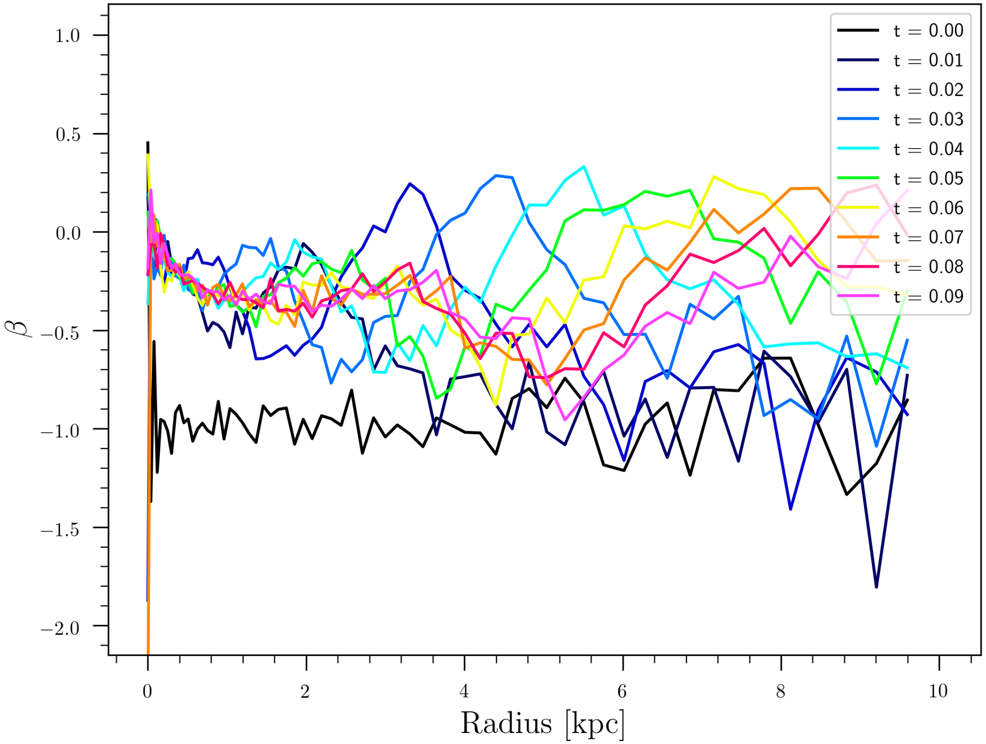
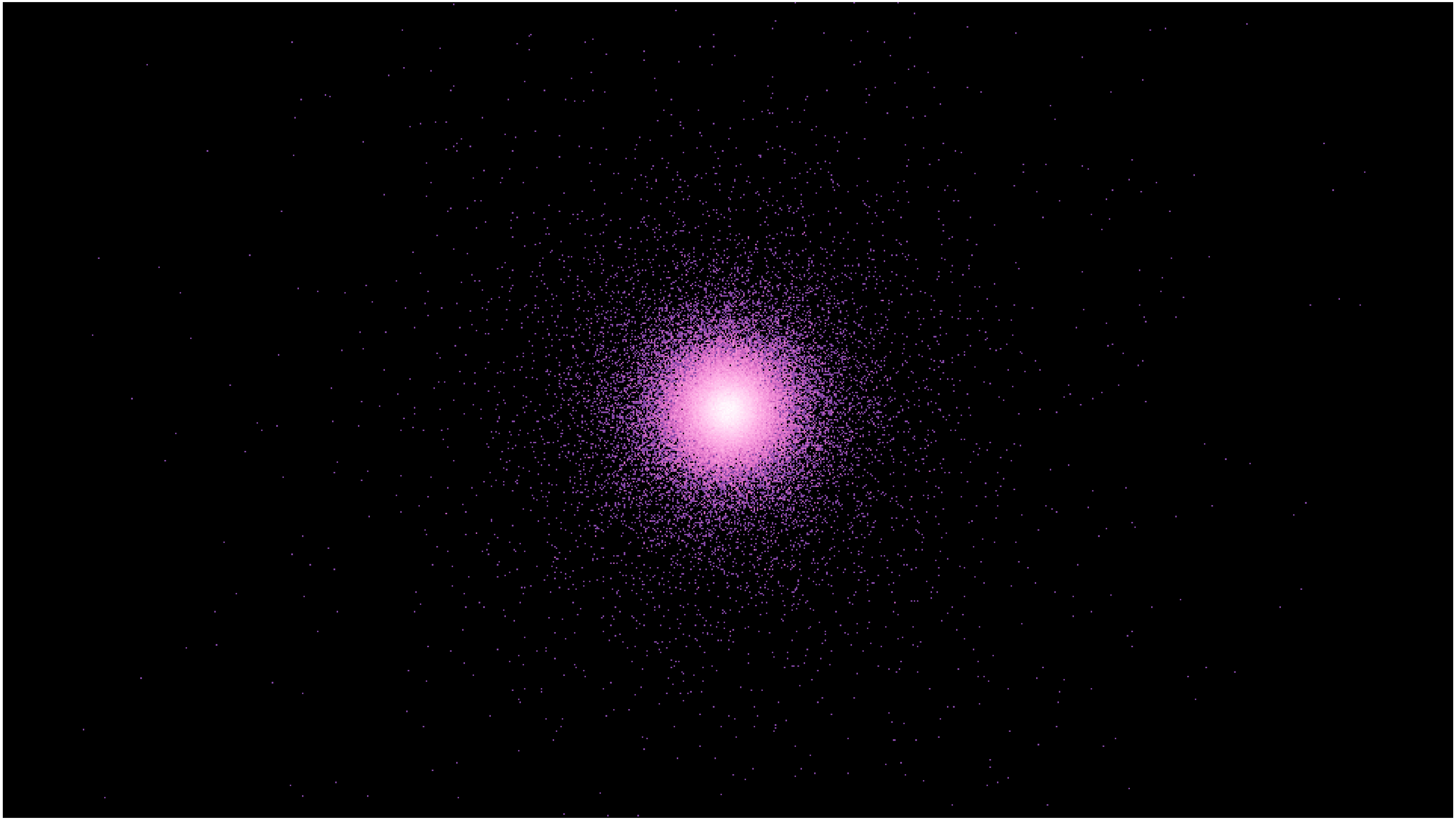




Plummer model with intermediate anisotropy $\beta = 0$ (ergodic)



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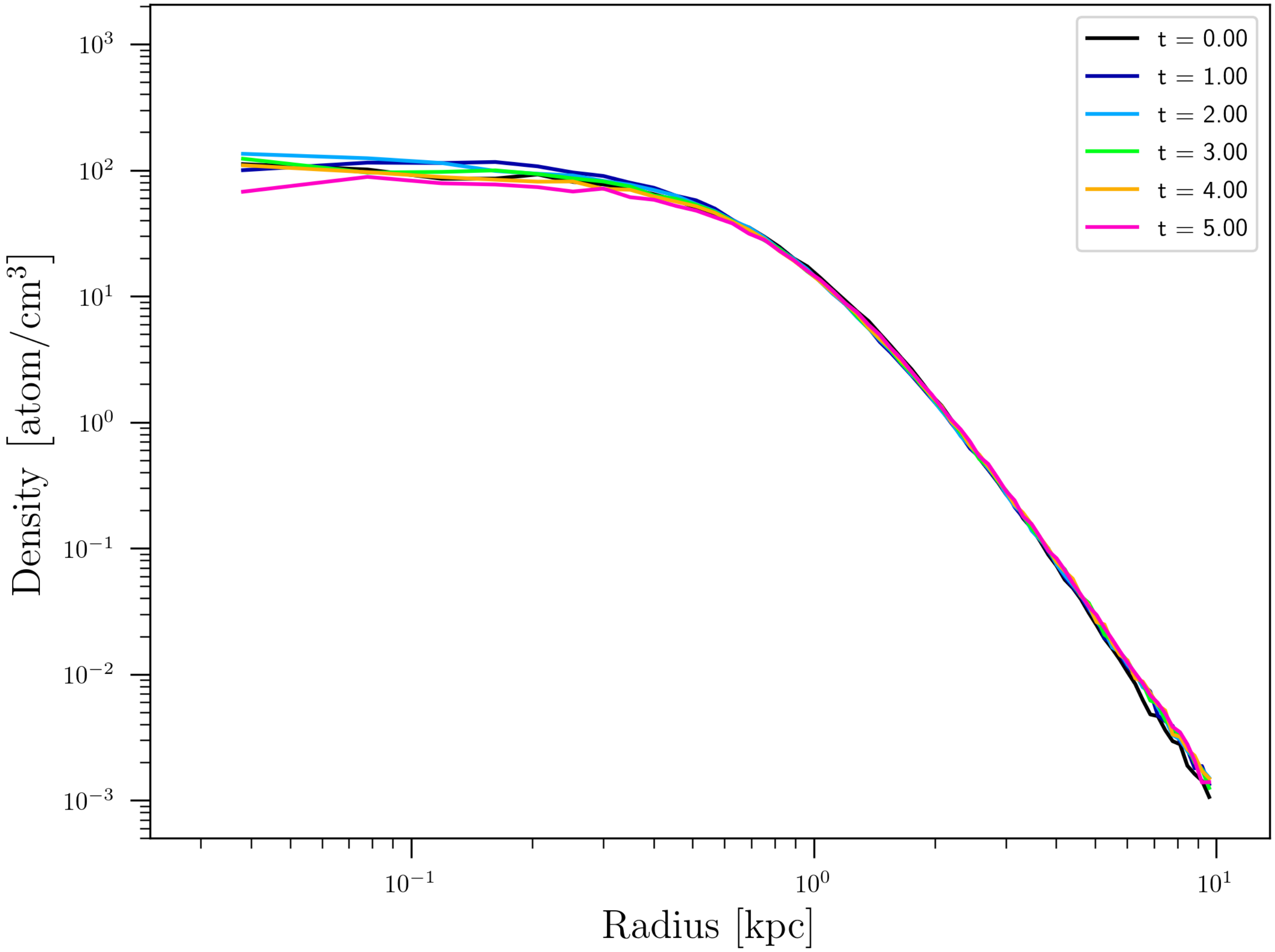

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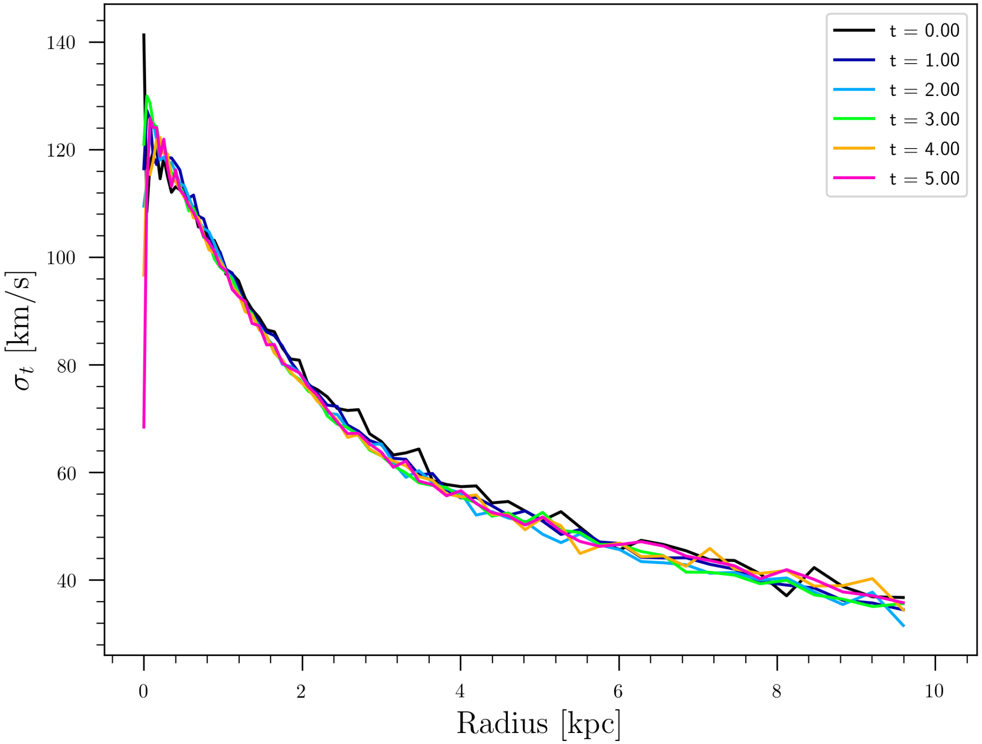

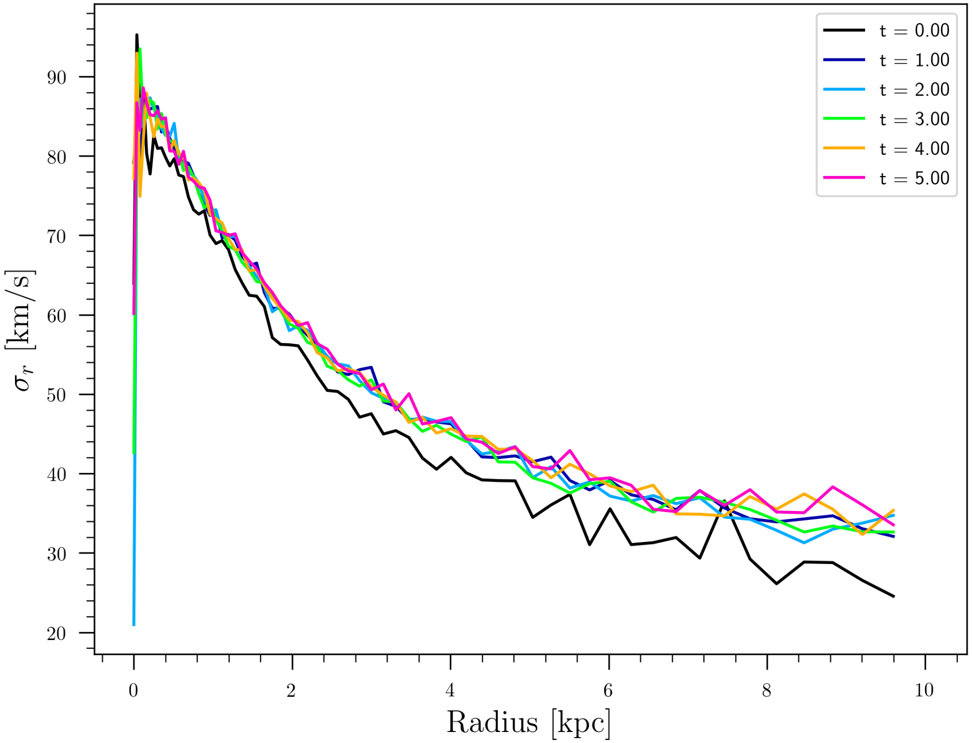

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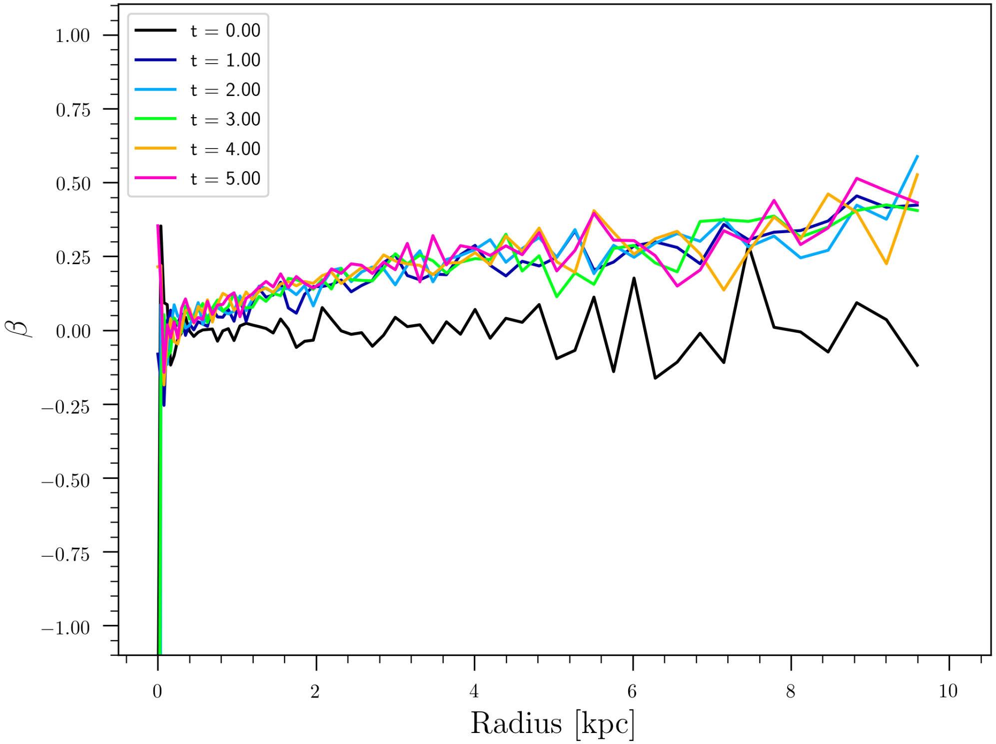
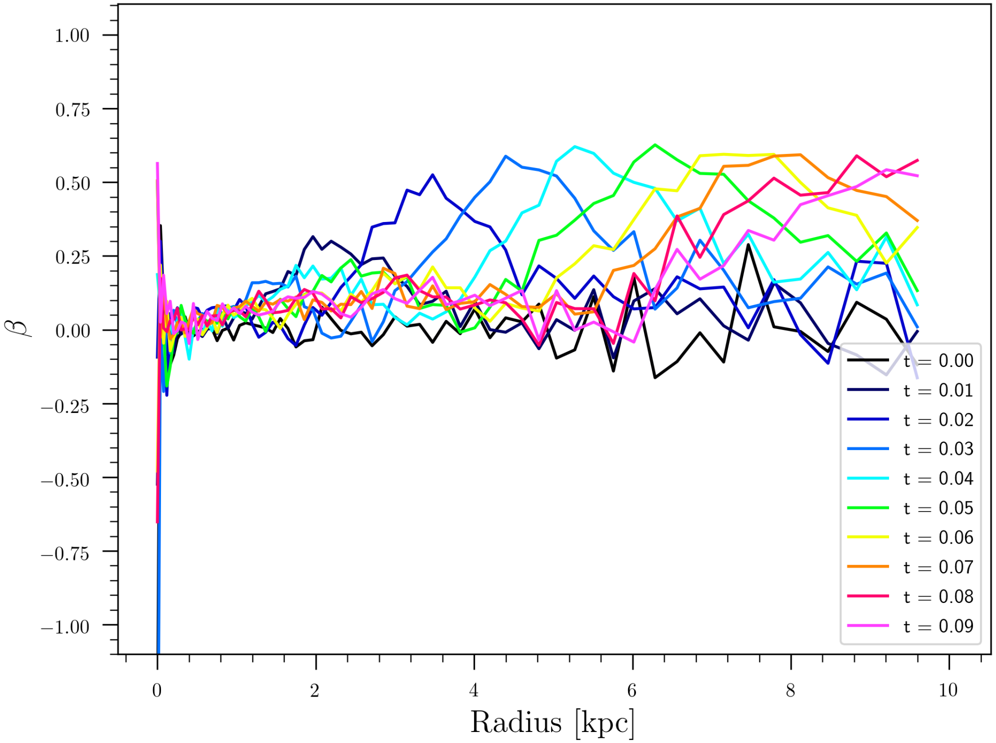
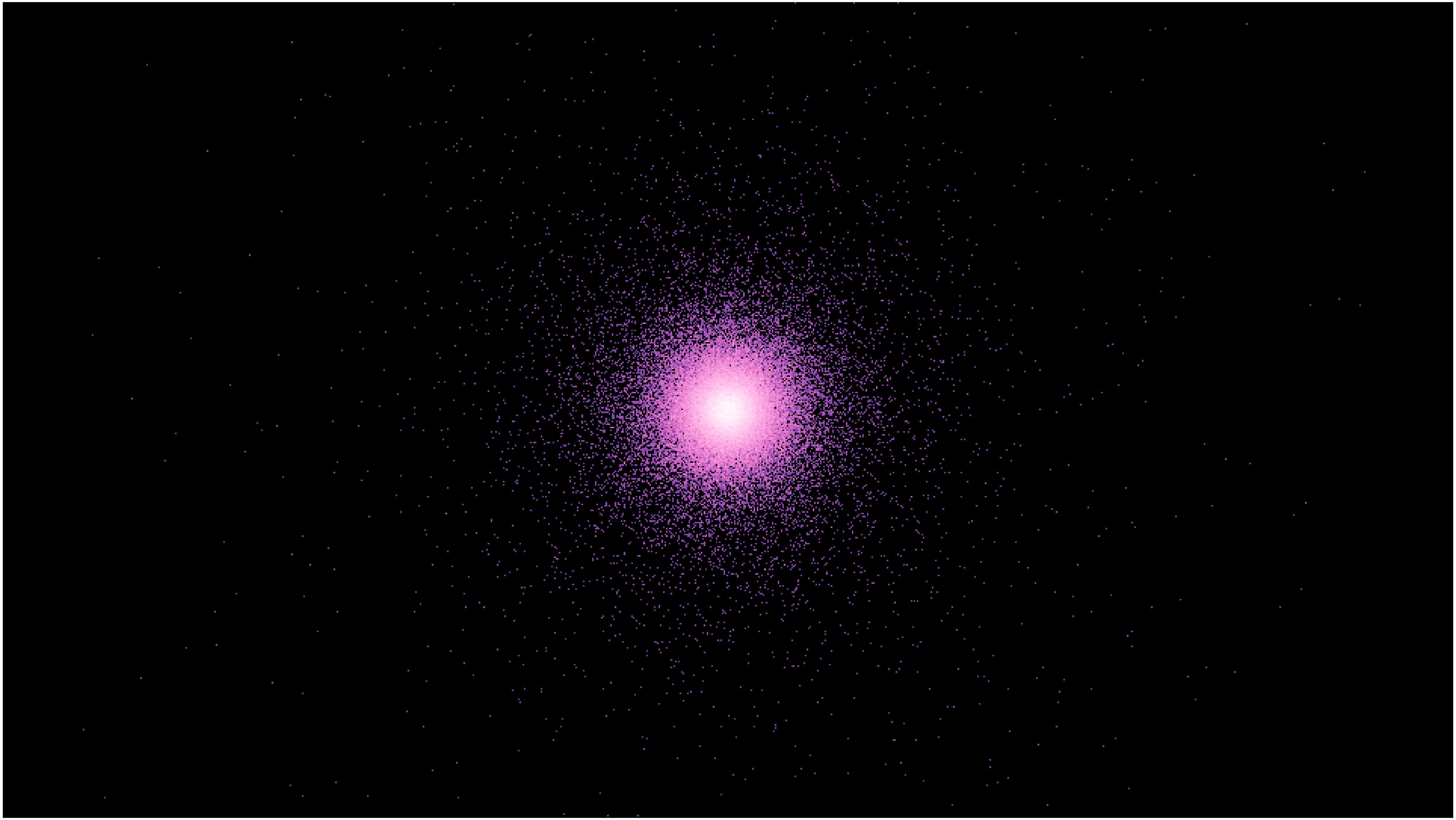




Plummer model with intermediate anisotropy $\beta = 0.875$



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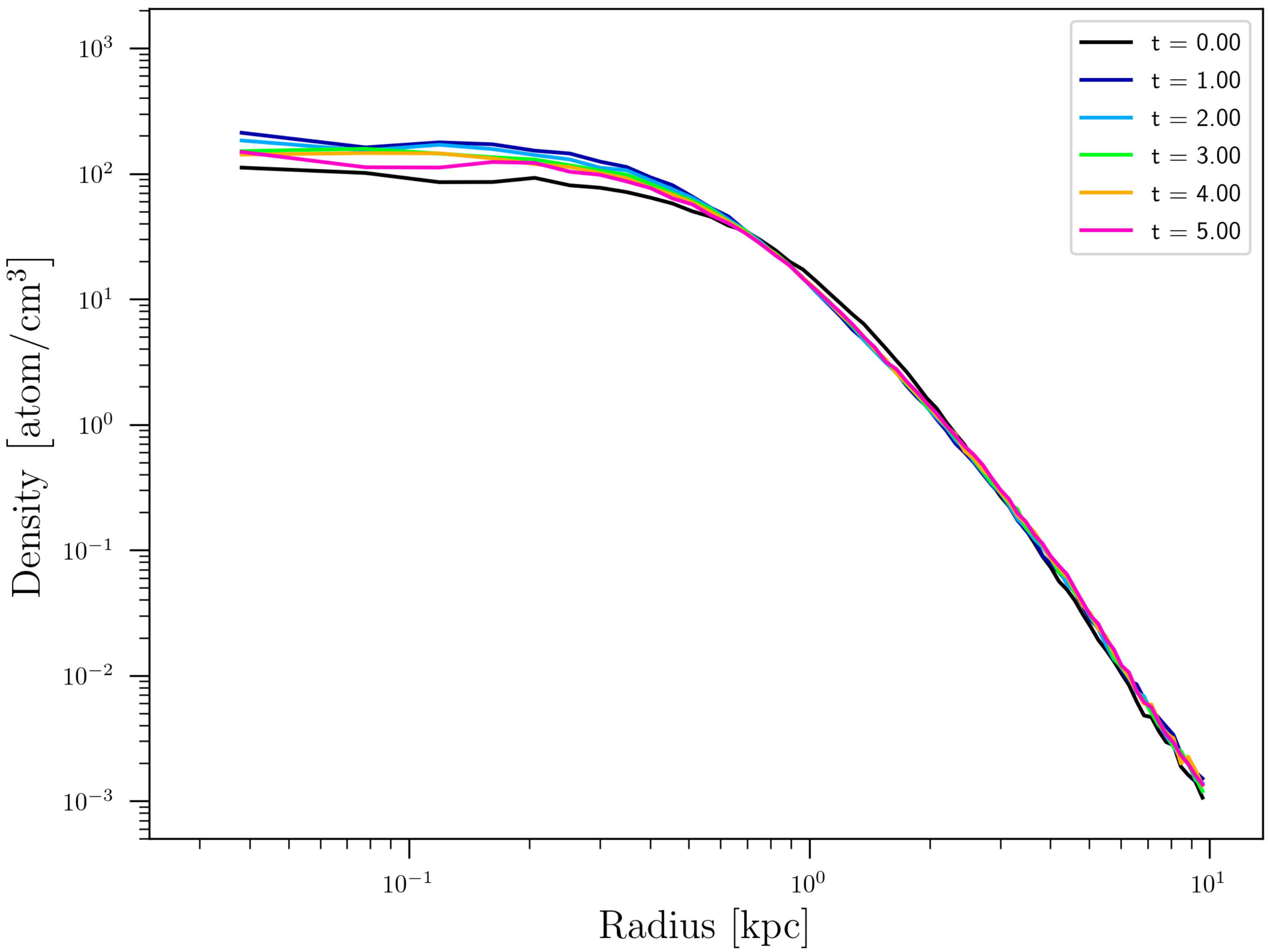

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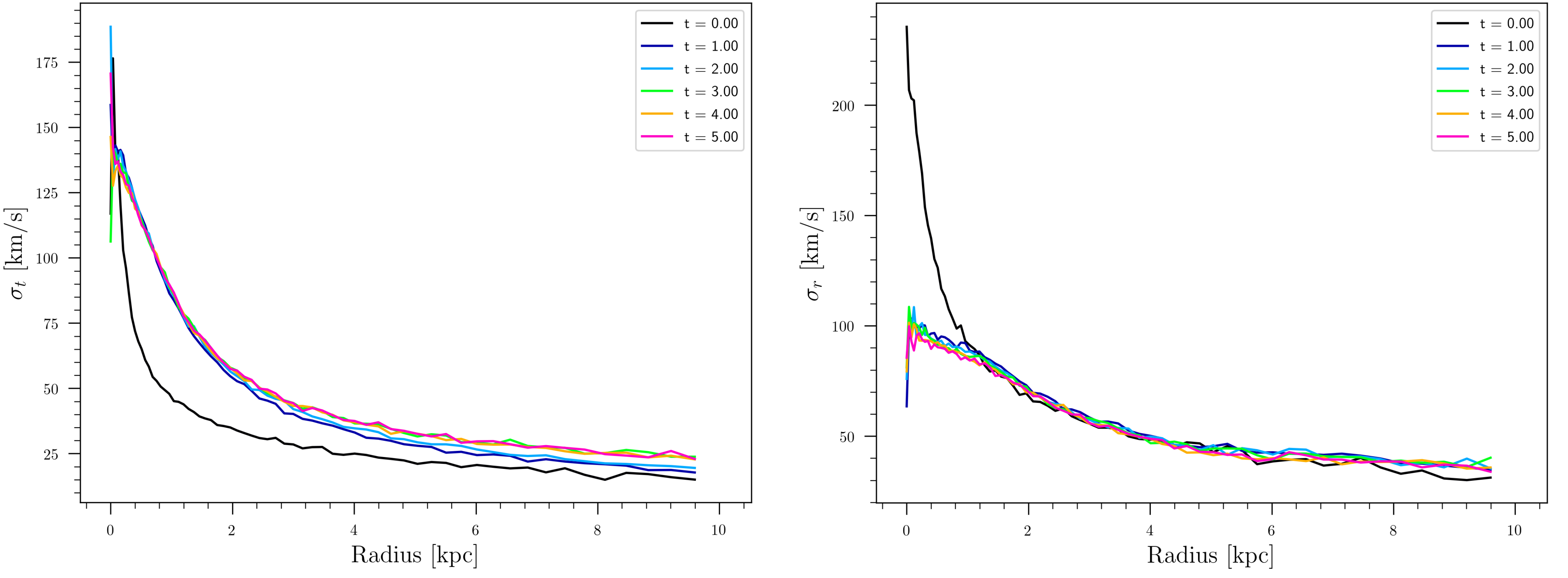
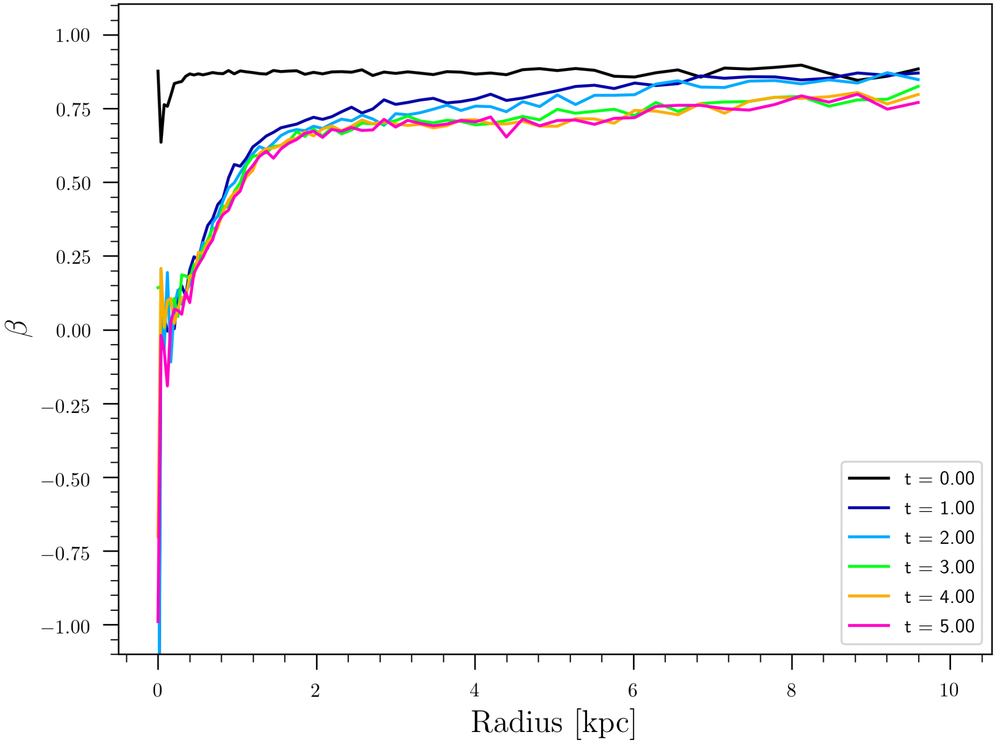
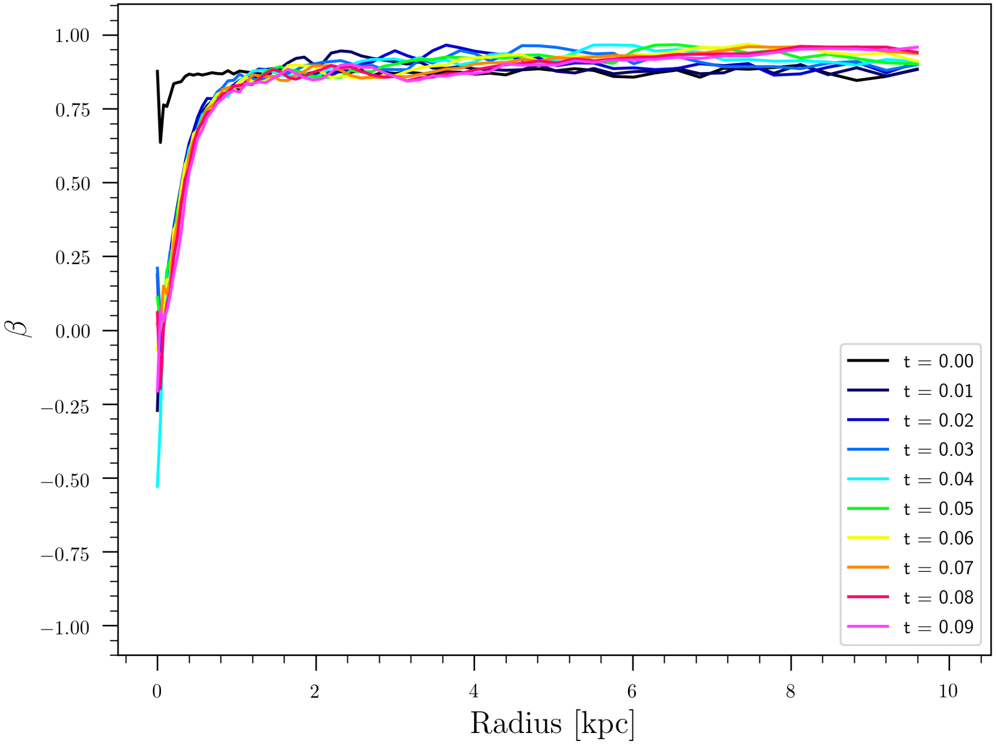


Plummer model with intermediate anisotropy $\beta = 0.875$

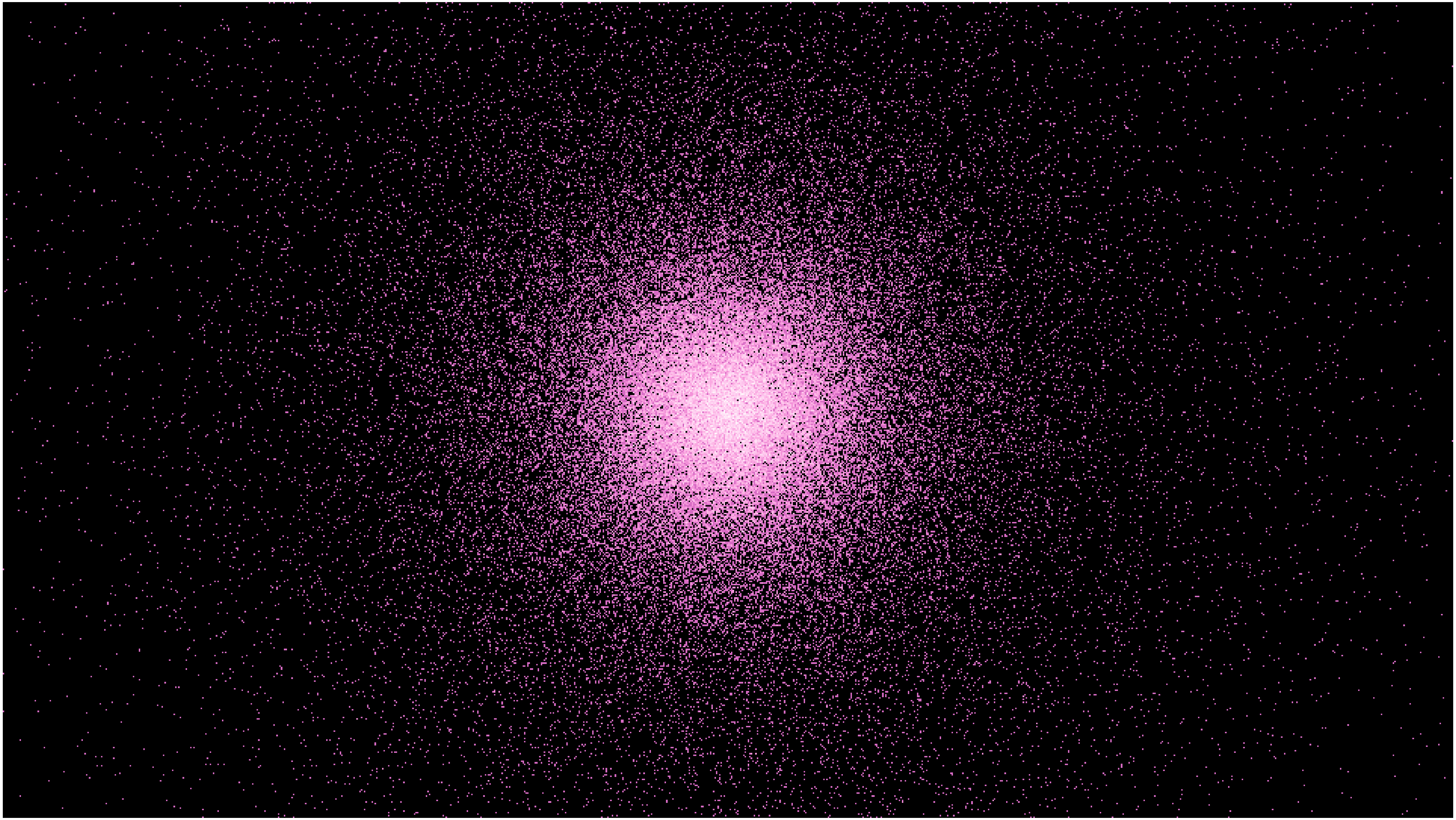




Miyamoto-Nagai razor-thin disk

cold disk: $\sigma_R = \sigma_\phi = 0$

Not self-gravitating !



Miyamoto-Nagai razor-thin disk

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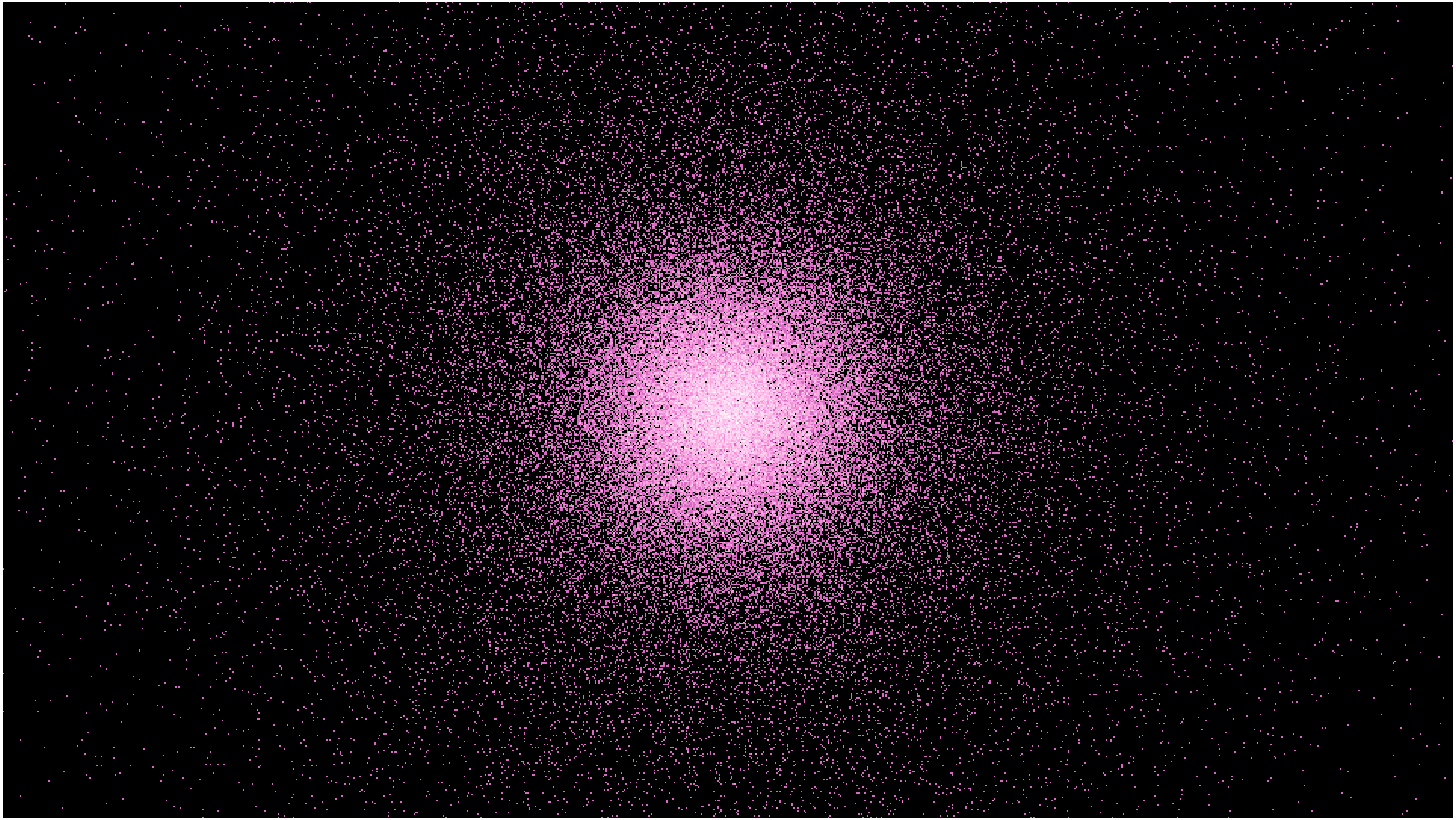
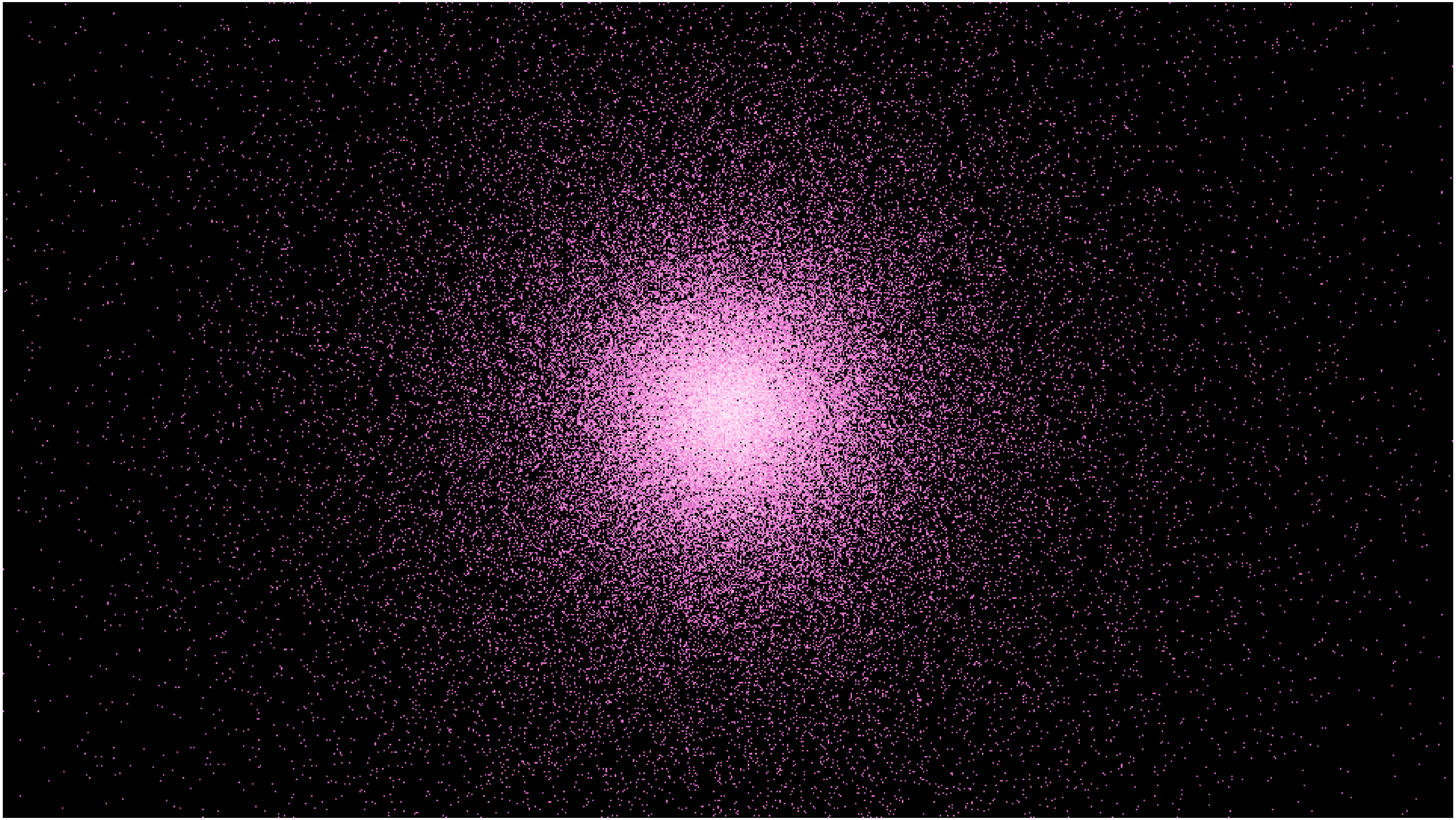


Miyamoto-Nagai razor-thin disk (counter-rotating !)

$$\sigma_R = 0 \ \sigma_\phi \neq 0$$

Not self-gravitating !



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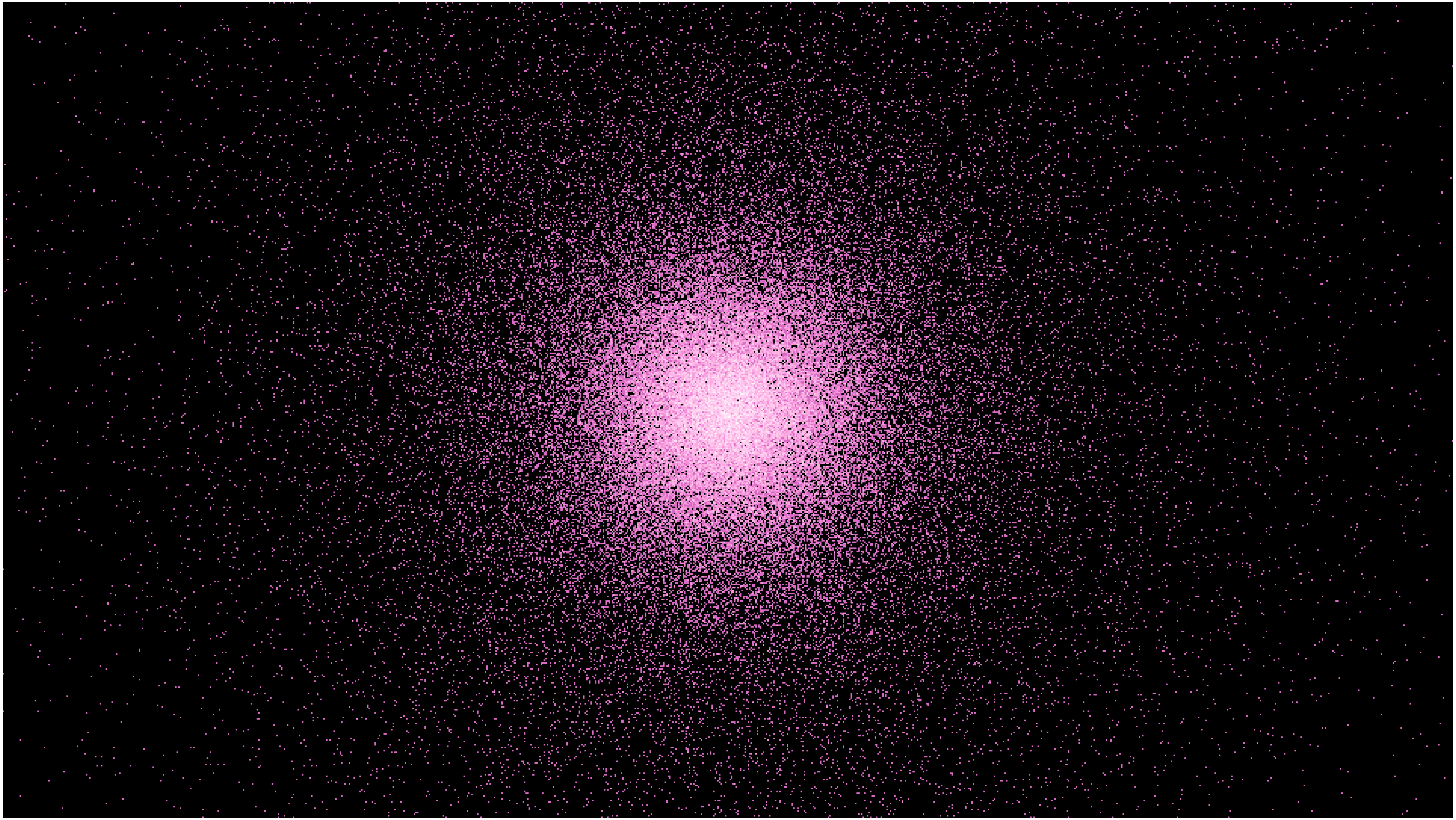
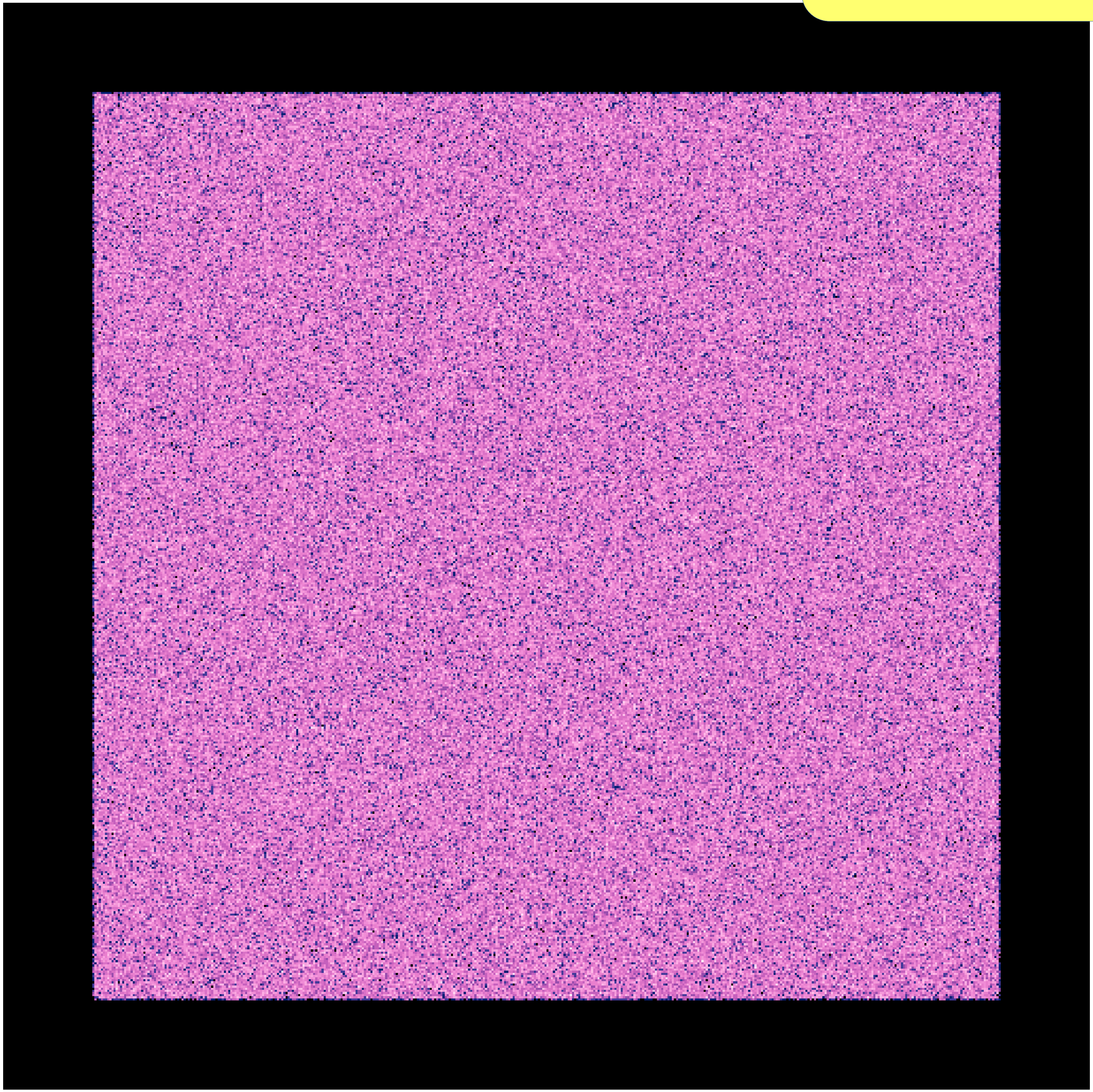


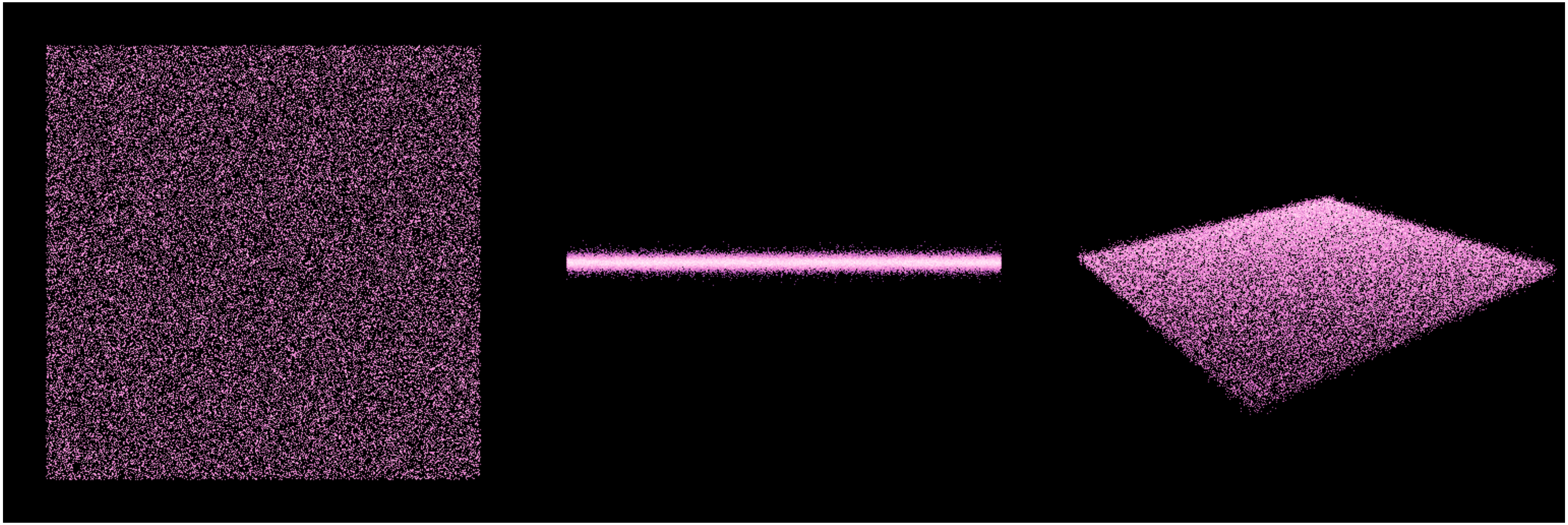
Self-gravitating infinite slab of infinite thickness

Not kinetic energy !



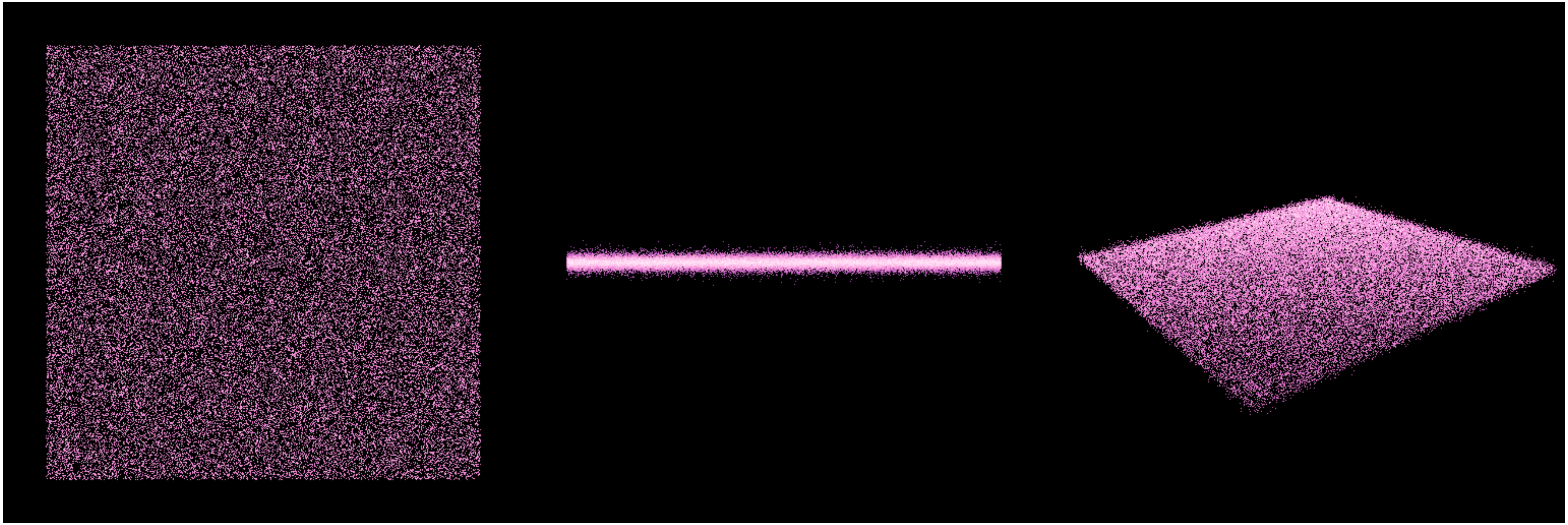
Self-gravitating infinite slab of finite thickness

$$\sigma_x = \sigma_y > \sigma_z$$



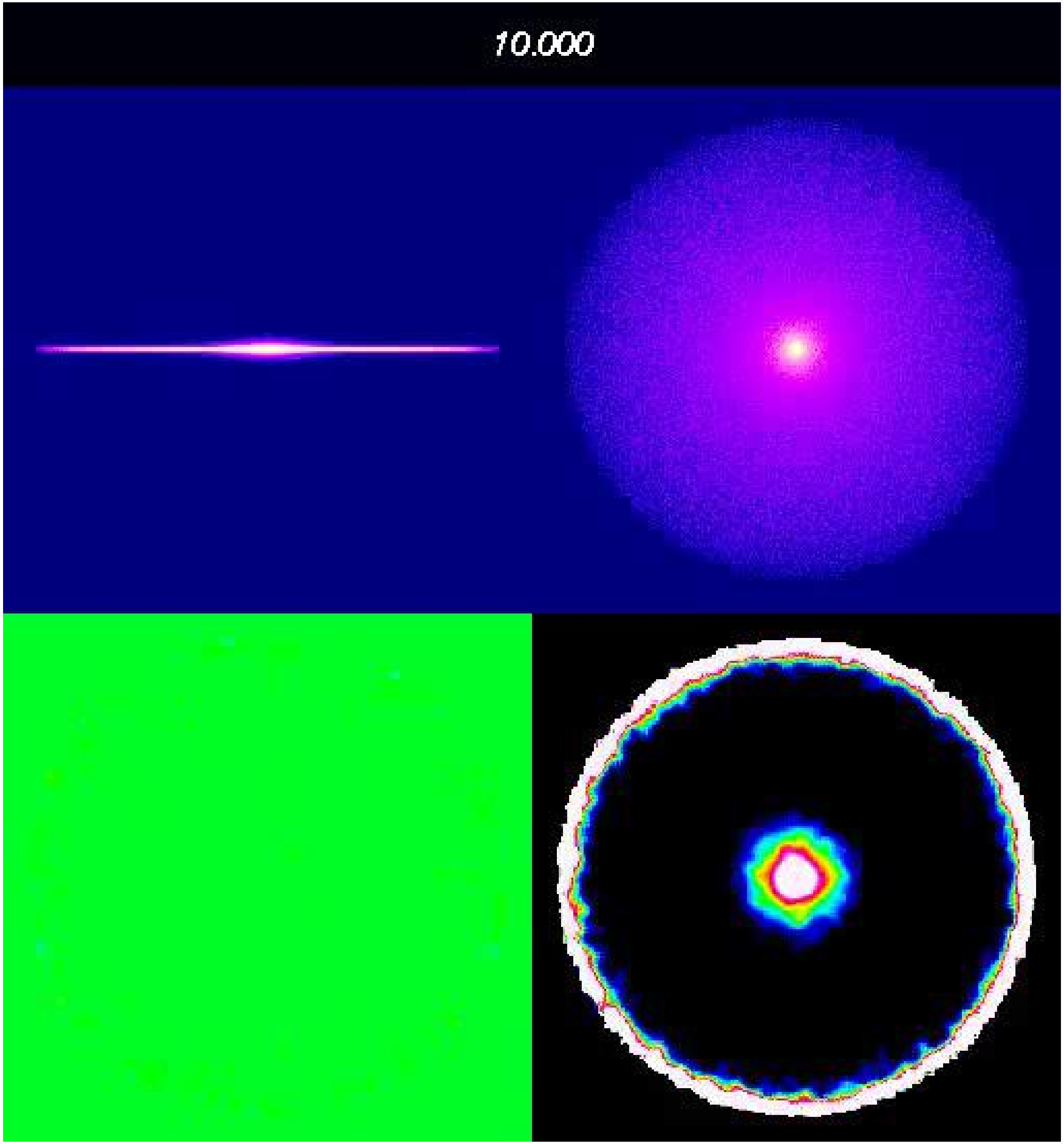
Self-gravitating infinite slab of finite thickness

$$\sigma_x = \sigma_z = 0$$



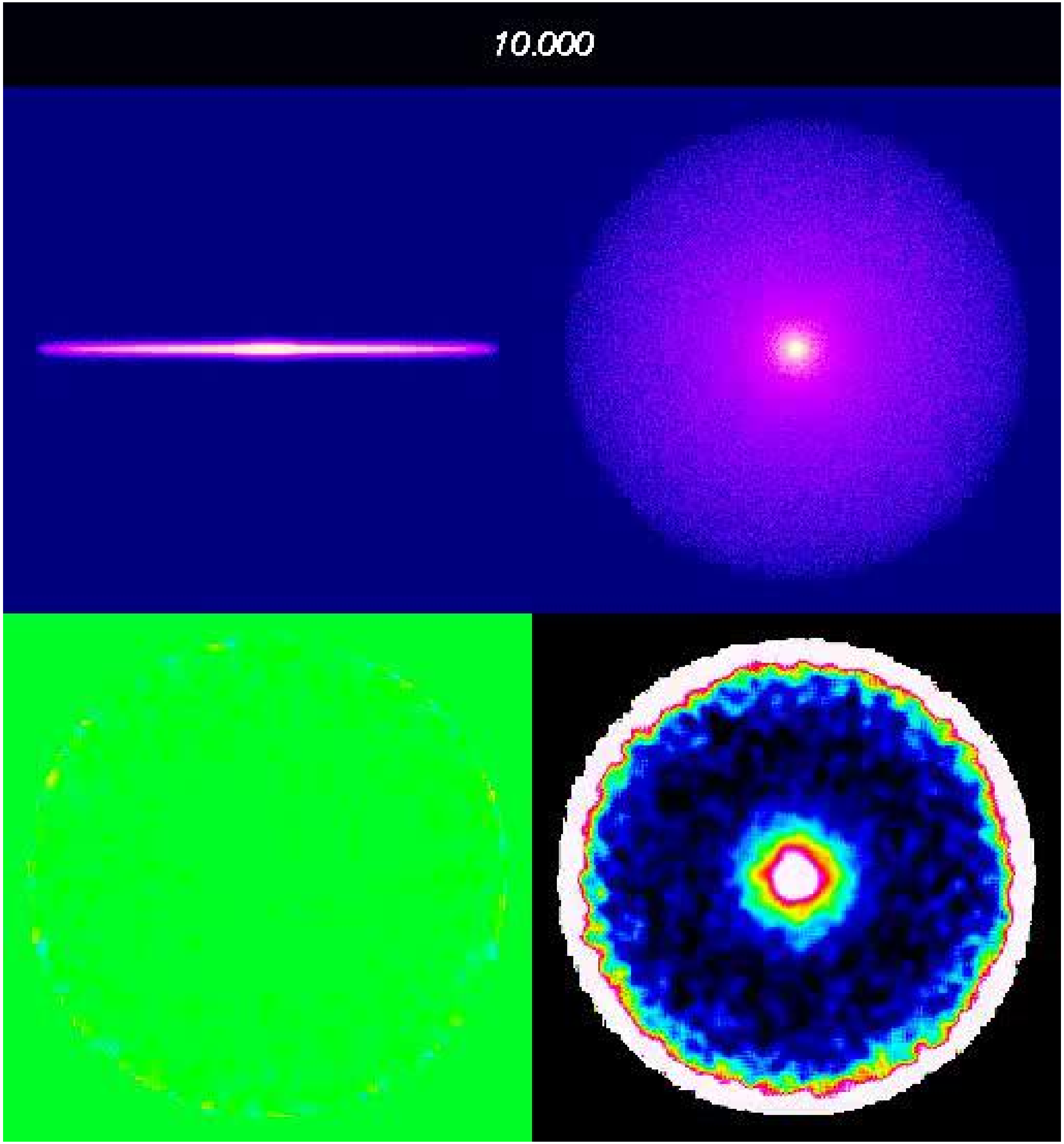
Warped disks

$$\sigma_z/\sigma_r = 0.18$$



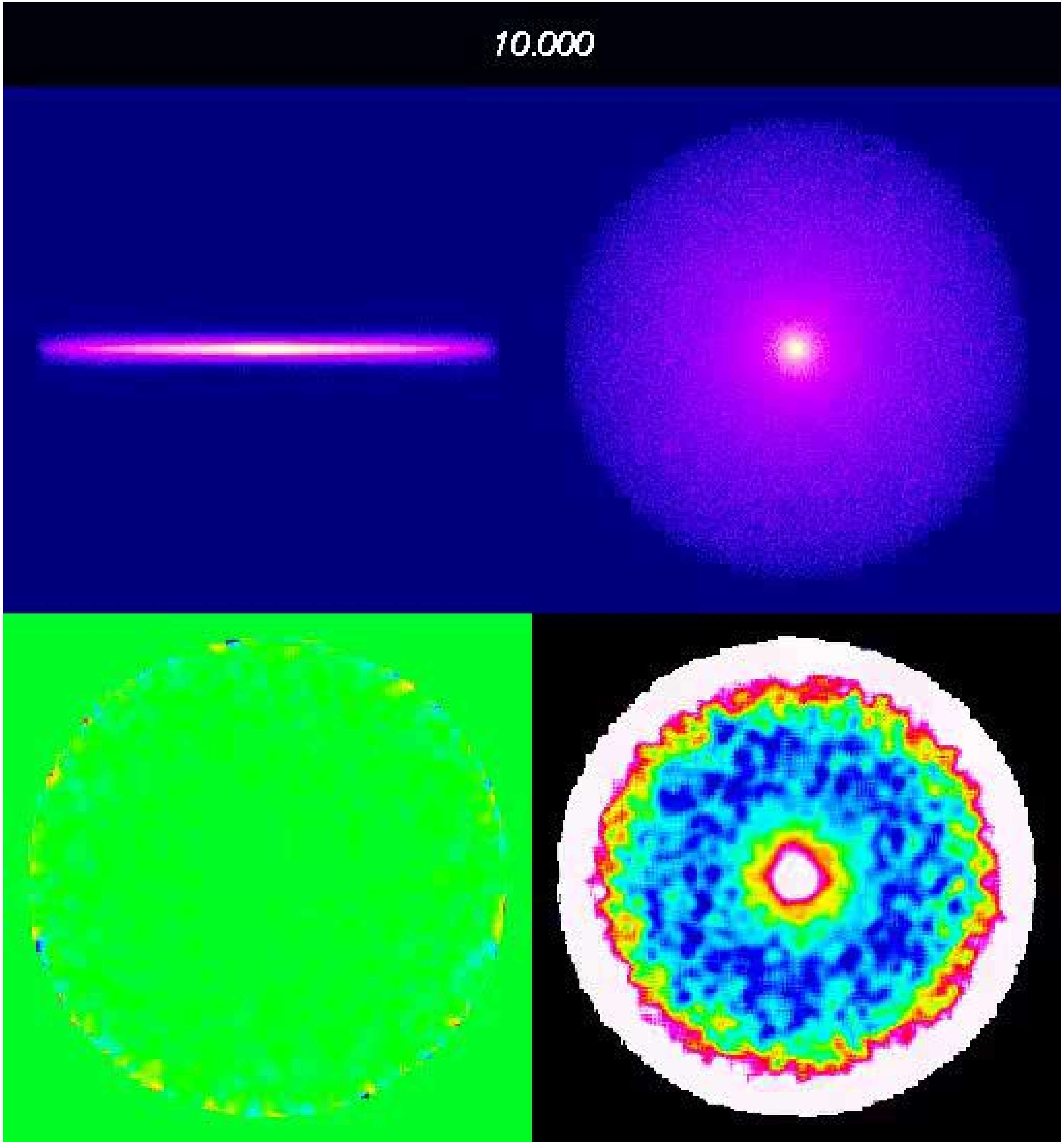
Warped disks

$$\sigma_z/\sigma_r = 0.21$$



Warped disks

$$\sigma_z/\sigma_r = 0.25$$

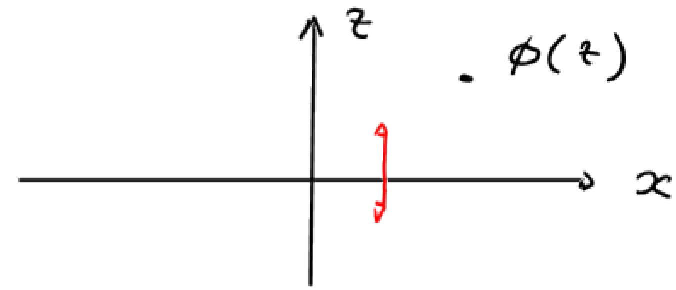


ESO 510-g13

Comments on the N-body experiments

- Stellar systems at equilibrium are not necessary stable !

What is the origin of the instability ?



- **Ingredients**

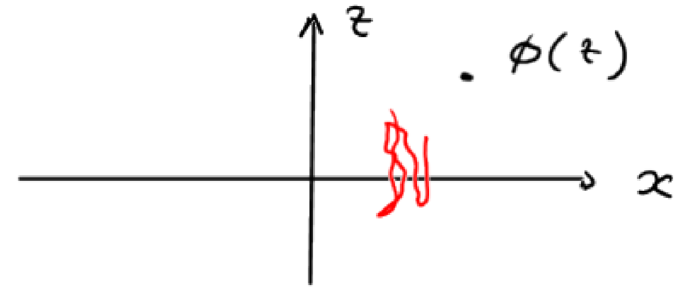
① Break of symmetry

Example: ideal infinite slab $\Rightarrow$ 3 uncoupled motions

$$\begin{cases} H_x = \frac{1}{2}\dot{x}^2 \\ H_y = \frac{1}{2}\dot{y}^2 \\ H_z = \frac{1}{2}\dot{z}^2 + \phi(z) \end{cases} \qquad \begin{cases} \ddot{x} = 0 \\ \ddot{y} = 0 \\ \ddot{z} = -\dfrac{\partial \phi}{\partial z} \end{cases}$$

3 integrals of motion

What is the origin of the instability ?



- Ingredients

(1) Break of symmetry

Example: ideal infinite slab $\Rightarrow$ ~~3 uncoupled motions~~

$$H = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + \frac{1}{2}\dot{z}^2 + \phi(x,y,z)$$

$$\begin{cases} \ddot{x} = -\dfrac{\partial \phi}{\partial x} \\ \ddot{y} = -\dfrac{\partial \phi}{\partial y} \\ \ddot{z} = -\dfrac{\partial \phi}{\partial z} \end{cases}$$

⚠ Noise induced by the numerical discretisation generates coupling terms

$$\phi(z) \Rightarrow \phi(x,y,z)$$

The equations becomes coupled, the accessible phase space is larger (1 integral of motion)

What is the origin of the instability ?

- Ingredients

(2) Anisotropies in the velocity dispertions

- systems with strong anisotropies are strongly unstable

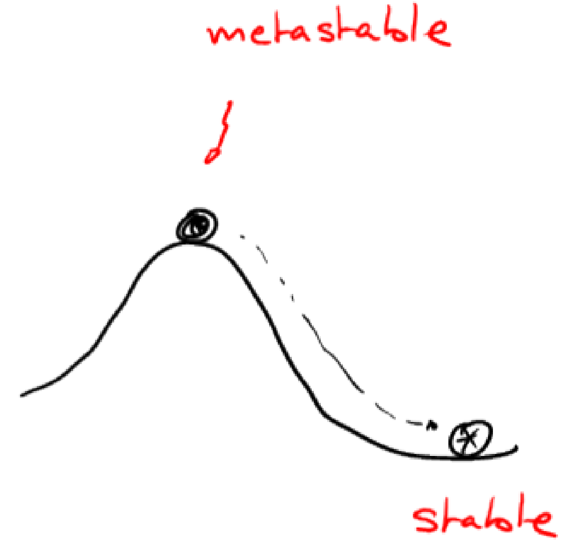
(3) Dynamically "cold" systems

- low velocity dispertions (σ^2)

$\equiv$ lower kinetic energy (K)

What is the origin of the instability?

- "Motivation"



- The system (at equilibrium) wants to change its state

  move to a "more probable" state ?

It can do so if :

1. The initial state has a low probability (anisotropic velocity dispersions)
2. It has some freedom to access other parts of the phase space (break of symmetry)
3. The new state guarantee the conservation of integrals of motion $(E, \vec{L})$

Is there a link with the notion of entropy as defined in thermodynamics ?

- could the change of state observed be related to an increase of entropy ?

Principle of maximum entropy:

S : entropy (see Landau & Lifshitz, Statistical physics)

$$S := -\int \rho \ln(\rho)$$

ρ : probability density in the phase space

↳ integral over the phase space

Thermodynamics second law

If the system is isolated, S increases up to a maximum.
The, DF is then the most probable one, the one that maximizes S.

Entropy of a collisionless system of N particles

By analogy : $p \rightarrow f(\vec{q},\vec{p})$

$$S := -N \int d\vec{q}\, d\vec{p}\; f(\vec{q},\vec{p}) \ln\left(f(\vec{q},\vec{p}) \right)$$

f : DF

Can we maximize S for a system of a given mass M and energy E ?

Yes : The solution is the isothermal sphere !

But ! The isothermal sphere is unphysical

$$\begin{cases} M = \infty \\ E = \infty \end{cases}$$

Where is the problem ?

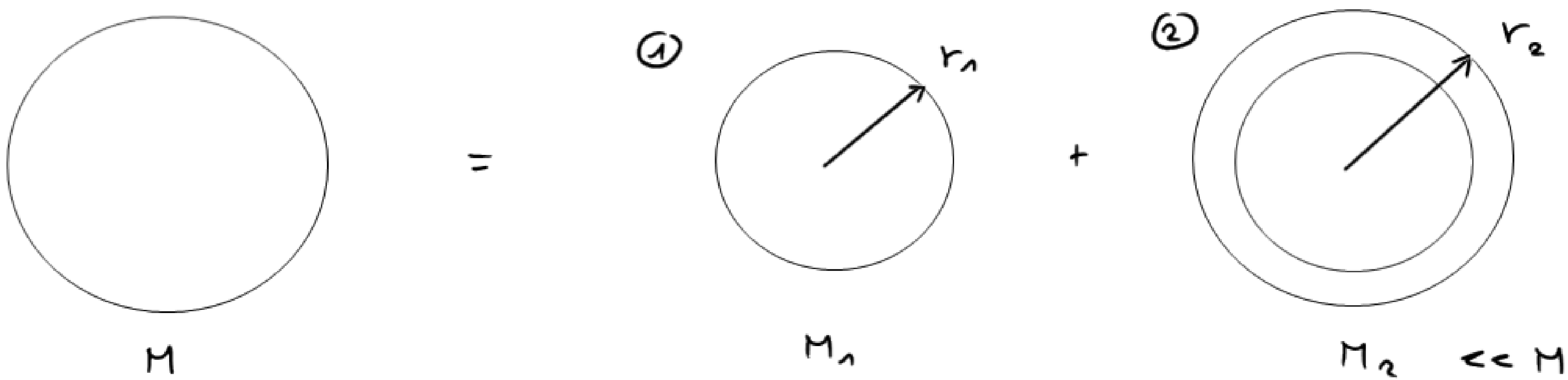
Corollary : No DF with finite mass M and finite energy E maximizes S

Demonstration

Spherical system $\left\{ \begin{array}{ll} \text{mass} & M \\ \text{total energy} & E \end{array} \right.$

Idea: split the system into : ① inner part

② outer part



Virial Equation

$2K + W = 0$

$E = K + W = \frac{W}{2}$

$E_1 = \frac{W}{2} \simeq -\frac{GM_1^2}{2r_1}$

virial

$E_2 \simeq -\frac{GM_1 M_2}{2r_2}$

the pot. is dominated by M_1

Idea :

$\varepsilon \ll 1$

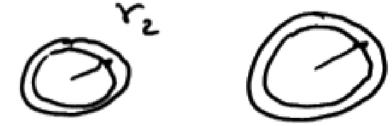
$r_1 \longrightarrow (1-\varepsilon)\, r_1$

- **Shrink the inner sphere (1)**

→ assume (1) to reach the virial equilibrium

$$E_1 \longrightarrow E_1' = -\frac{G M_1^2}{2 r_1'} = -\frac{G M_1^2}{2 r_1 (1-\varepsilon)}$$

$$\Delta E = E_1 - E_1' \cong \varepsilon \frac{G M_1^2}{r_1} \qquad (>0 : \text{excess of grav. nrj})$$



- **Inject ΔE in the outer shell (2)**

→ assume (2) to reach the virial equilibrium

$$r_2 \longrightarrow r_2', \quad E_2 \longrightarrow E_2' = E_2 + \Delta E = -\frac{G M_1 M_2}{r_2'}$$

$$r_2' > r_2$$

$$\frac{1}{r_2'} \sim E_2 + \Delta E$$

Estimation of the entropy of ②

$$S_2 \sim -N_2 \underbrace{\int d^3x\, d^3v\, f_2(x,v)}_{\simeq 1 \text{ (mass is conserved)}} \ln f_2(x,v)$$

density

$$f_2(x,v) \sim \frac{1}{V_2}$$

V_2 = volume of the phase space occupied by ②

virial

$$V_2 \sim \sigma_2'^3 \, r_2'^3 = \left(\frac{GM_1}{r_2'}\right)^{3/2} r_2'^3$$

$$\sim \left(GM_1 r_2'\right)^{3/2}$$

$$S_2 \sim \frac{3}{2} \ln(r_2') \sim -\frac{3}{2} \ln(E_2 + \Delta E)$$

$$S_2 \sim -\frac{3}{2} \ln \left(\underbrace{E_2}_{<0} + \underbrace{\Delta E}_{>0} \right)$$

if $\Delta E = -E_2$ $\left(\text{i.e } \varepsilon = \frac{1}{2} \frac{M_2}{M_1} \frac{r_1}{r_2} \right)$

$S_2 \longrightarrow \infty$!!!

$S_1 + S_2 \longrightarrow \infty$!!!

We can always collapse a portion of a system and release gravitational energy to the outer part (diffuse envelope) in such a way that the entropy increase (no bounds)

Conclusions

Stellar systems may increase they entropy, but never reach a maximum, as this maximum does not exist.

Finite stellar systems cannot reach a thermodynamical equilibrium.

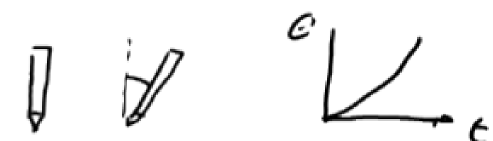
Our goal Study the stability of systems at equilibrium

Method : perturbation theory

perturbation → **response**

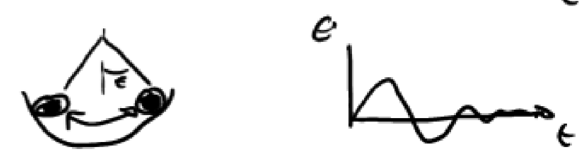
Types of responses

- Exponential growth of the perturbation



- Oscillation of the perturbation

- Die of the perturbation



The End