Exercise Set 11: Solution
Quantum Computation

Exercise 1 *QM reminder and 3-qubit repetition code*

(a) Let us start with Z, which is the simplest: its two eigenvectors are $|0\rangle$ and $|1\rangle$, with respective eigenvalues $+1$ and -1. For X, observe that

$$X(|0\rangle + |1\rangle) = |1\rangle + |0\rangle \quad \text{and} \quad X(|0\rangle - |1\rangle) = |1\rangle - |0\rangle$$

so the two (normalized) eigenvectors of X are $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $|-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$, again with respective eigenvalues $+1$ and -1.

(b) Let us compute

$$[X, Z] = XZ - ZX = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$

and

$$\{X, Z\} = XZ + ZX = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

so $XZ = -ZX$.

(c) If $i = j$, then clearly $[Z_i, Z_i] = 0$ and if $i \neq j$, then for example in the case $i = 1$ and $j = 2$, for any x_1, x_2, x_3, we obtain

$$\begin{aligned} [Z_1, Z_2] |x_1, x_2, x_3\rangle &= (Z_1 Z_2 - Z_2 Z_1) |x_1, x_2, x_3\rangle \\ &= (-1)^{x_1+x_2} |x_1, x_2, x_3\rangle - (-1)^{x_1+x_2} |x_1, x_2, x_3\rangle = 0 \end{aligned}$$

so $[Z_i, Z_j] = 0$ for all i, j. For the commutators between the X's and the Z's, we obtain for $i = j(= 1$, for example):

$$\begin{aligned} [X_1, Z_1] |x_1, x_2, x_3\rangle &= (X_1 Z_1 - Z_1 X_1) |x_1, x_2, x_3\rangle \\ &= (-1)^{x_1} |\overline{x}_1, x_2, x_3\rangle - (-1)^{\overline{x}_1} |\overline{x}_1, x_2, x_3\rangle = 2(-1)^{x_1} |\overline{x}_1, x_2, x_3\rangle \end{aligned}$$

If $i \neq j$ ($i = 1$ and $j = 2$ for example), then we obtain

$$\begin{aligned} [X_1, Z_2] |x_1, x_2, x_3\rangle &= (X_1 Z_2 - Z_2 X_1) |x_1, x_2, x_3\rangle \\ &= (-1)^{x_2} |\overline{x}_1, x_2, x_3\rangle - (-1)^{x_2} |\overline{x}_1, x_2, x_3\rangle = 0 \end{aligned}$$

so $[X_i, Z_j] = 0$ for all $i \neq j$.

(d) The order does not matter here, as the operators A_1 and A_2 commute. Besides, $|\psi_1\rangle$ is an eigenvector of both A_1 and A_2 in any case.

(e) If there is no bit flip (i.e., $|\psi_1\rangle = |\psi_0\rangle$), then $A_1 |\psi_1\rangle = (+1) |\psi_1\rangle$ and $A_2 |\psi_1\rangle = (+1) |\psi_1\rangle$, in which case no X operation is needed.

If the first bit is flipped (i.e., $|\psi_1\rangle = \alpha |100\rangle + \beta |011\rangle$), then $A_1 |\psi_1\rangle = (+1) |\psi_1\rangle$ and $A_2 |\psi_1\rangle = (-1) |\psi_1\rangle$; in this case, the observable X_1 must then be measured.

If the second bit is flipped (i.e., $|\psi_1\rangle = \alpha |010\rangle + \beta |101\rangle$), then $A_1 |\psi_1\rangle = (-1) |\psi_1\rangle$ and $A_2 |\psi_1\rangle = (+1) |\psi_1\rangle$; in this case, the observable X_2 must then be measured.

If the third bit is flipped (i.e., $|\psi_1\rangle = \alpha |001\rangle + \beta |110\rangle$), then $A_1 |\psi_1\rangle = (-1) |\psi_1\rangle$ and $A_2 |\psi_1\rangle = (-1) |\psi_1\rangle$; in this case, the observable X_3 must then be measured.

(f) This is not a problem, as the (potential) measurement of X_i comes clearly *after* the measurements of A_1 and A_2. But note that $|\psi_1\rangle$ is *not* an eigenvector of X_i.

(g) No, as the measurements of A_1 and A_2 suffice to determine the position of the bit-flip; an additional measurement of A_3 would therefore be useless (but not detrimental).

Exercise 2 *The Shor code*

Let us compute the states after the successive stages for an input $|\psi\rangle =$ either $|0\rangle$ or $|1\rangle$ (the final result is then obtained by linear superposition):

$$|\psi_0\rangle = |\psi\rangle \otimes |0\rangle^{\otimes 8}$$

$$|\psi_1\rangle = |\psi\rangle \otimes |0\rangle^{\otimes 2} \otimes |\psi\rangle \otimes |0\rangle^{\otimes 2} \otimes |\psi\rangle \otimes |0\rangle^{\otimes 2}$$

$$|\psi_2\rangle = \left(\frac{(-1)^\psi |\psi\rangle + |\overline{\psi}\rangle}{\sqrt{2}}\right) \otimes |0\rangle^{\otimes 2} \otimes \left(\frac{(-1)^\psi |\psi\rangle + |\overline{\psi}\rangle}{\sqrt{2}}\right) \otimes |0\rangle^{\otimes 2} \otimes \left(\frac{(-1)^\psi |\psi\rangle + |\overline{\psi}\rangle}{\sqrt{2}}\right) \otimes |0\rangle^{\otimes 2}$$

$$|\psi_3\rangle = \left(\frac{(-1)^\psi |\psi,\psi,\psi\rangle + |\overline{\psi},\overline{\psi},\overline{\psi}\rangle}{\sqrt{2}}\right) \otimes \left(\frac{(-1)^\psi |\psi,\psi,\psi\rangle + |\overline{\psi},\overline{\psi},\overline{\psi}\rangle}{\sqrt{2}}\right) \otimes \left(\frac{(-1)^\psi |\psi,\psi,\psi\rangle + |\overline{\psi},\overline{\psi},\overline{\psi}\rangle}{\sqrt{2}}\right)$$

which is indeed the proposed encoding.