Exercise Set 12
Quantum Computation

Exercise 1 *QM reminder and 3-qubit repetition code*

Here is a first important fact in quantum mechanics:

Let A be an observable (= self-adjoint matrix) acting on some finite N-dimensional Hilbert space $\mathcal{H}$. Because A is self-adjoint, its eigenvectors ($|\psi_i\rangle$, $1 \leq i \leq N$) form an orthonormal basis of $\mathcal{H}$; let also (λ_i, $1 \leq i \leq N$) denote the corresponding eigenvalues of A.

Assume the current state of the system is $|\psi\rangle \in \mathcal{H}$. When one performs a measurement of A in state $|\psi\rangle$, then with probability $P(i) = |\langle\psi_i|\psi\rangle|^2$, the outcome state is $|\psi_i\rangle$ and the corresponding measured value of A is the eigenvalue λ_i. Note importantly that if $|\psi\rangle$ is already an eigenvector of A, then it is left unchanged.

Let now $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. These are observables acting on $\mathbb{C}^2$ and satisfying

$$X|x\rangle = |\overline{x}\rangle \quad \text{and} \quad Z|x\rangle = (-1)^x |x\rangle$$

for $|x\rangle \in \{|0\rangle, |1\rangle\}$ (the computational basis of $\mathbb{C}^2$).

(a) What are the eigenvalues and eigenvectors of both X and Z?

(b) Compute both the *commutator* and the *anti-commutator* of X and Z defined as

$$[X, Z] = XZ - ZX \quad \text{and} \quad \{X, Z\} = XZ + ZX$$

Consider now $\mathcal{H} = (\mathbb{C}^2)^3$, as well as the operators X_i and Z_j defined as

$$X_i |x_1, x_2, x_3\rangle = |y_1, y_2, y_3\rangle, \quad \text{where } y_i = \overline{x}_i \text{ and } y_j = x_j \text{ for all } j \neq i$$

and

$$Z_j |x_1, x_2, x_3\rangle = (-1)^{x_j} |x_1, x_2, x_3\rangle$$

Note: These are short-hand notations. The observable X_1 for example is strictly speaking equal to $X_1 \otimes I \otimes I$ (acting on $\mathcal{H} = (\mathbb{C}^2)^3$).

(c) Compute the commutators $[Z_i, Z_j]$ and $[X_i, Z_j]$, for $i, j \in \{1, 2, 3\}$.

Here is another important fact in quantum mechanics:

We say that two observables A_1, A_2 commute *if* $[A_1, A_2] = 0$. *If two observables commute, they can be measured without one measurement affecting another, regardless of the state.*

Consider now the following quantum repetition code with 3 qubits: the initial state $|\psi_0\rangle = \alpha\,|000\rangle + \beta\,|111\rangle$ (with $|\alpha|^2 + |\beta|^2 = 1$) experiences potentially a bit-flip to become state $|\psi_1\rangle$.

(d) The observables $A_1 = I \otimes Z_2 \otimes Z_3$ and $A_2 = Z_1 \otimes I \otimes Z_3$ are then successively measured in state $|\psi_1\rangle$. Does the order matter here? And is $|\psi_1\rangle$ an eigenvector of both A_1 and A_2?

(e) Given the outcomes of these two measurements, describe what measurement X_i comes next to correct the potential bit-flip.

(f) Is it a problem that the measurement X_i does not necessarily commute with the operators A_1 and A_2? And is $|\psi_1\rangle$ an eigenvector of X_i?

(g) In step (d), would a third measurement of the observable $A_3 = Z_1 \otimes Z_2 \otimes I$ be any helpful?

Exercise 2 *The Shor code*

The Shor code uses 9 qubits one qubit and corrects one error (a bit flip or a phase flip). The code length is 9, the code dimension is 2 (viewed as a subspace of the Hilbert space). Codewords are of the form $\alpha|0\rangle_S + \beta|1\rangle_S$ with

$$\frac{|000\rangle + |111\rangle}{\sqrt{2}} \otimes \frac{|000\rangle + |111\rangle}{\sqrt{2}} \otimes \frac{|000\rangle + |111\rangle}{\sqrt{2}} \equiv |0\rangle_S$$
$$\frac{|000\rangle - |111\rangle}{\sqrt{2}} \otimes \frac{|000\rangle - |111\rangle}{\sqrt{2}} \otimes \frac{|000\rangle - |111\rangle}{\sqrt{2}} \equiv |1\rangle_S$$

Verify that the following circuit realizes the proposed (unitary) encoding: