Exercise Set 9: Solution
Quantum Computation

Exercise 1 *Grover's algorithm for $N = 4$*

(a) The theory says that one query of the oracle in the quantum circuit suffices (as here, $M = 1 = N/4$). In other words, one "Grover operator" suffices.

(b) If P is any projector we have $(I-2P)(I-2P) = I - 4P + 4P^2 = I - 4P + 4P = I$. For the given U matrix this implies that $UU^\dagger = U^\dagger U = I$.

The entry $|00\rangle$ is mapped to

$$\begin{aligned}|00\rangle &\to |11\rangle \to \frac{1}{\sqrt{2}}(|10\rangle - |11\rangle) \to \frac{1}{\sqrt{2}}(|11\rangle - |10\rangle)\\ &\to \frac{1}{2}(|10\rangle - |11\rangle - |10\rangle - |11\rangle) = -|11\rangle \to -|00\rangle\end{aligned}$$

The entry $|10\rangle$ is mapped to

$$\begin{aligned}|10\rangle &\to |01\rangle \to \frac{1}{\sqrt{2}}(|00\rangle - |01\rangle) \to \frac{1}{\sqrt{2}}(|00\rangle - |01\rangle)\\ &\to \frac{1}{2}(|00\rangle + |01\rangle - |00\rangle + |01\rangle) = |01\rangle \to |10\rangle\end{aligned}$$

and we check also that $|01\rangle \to |01\rangle$ et $|11\rangle \to |11\rangle$.

(c) *Algorithmic steps:* We assume that $x_0 = 00$ without loss of generality.

1. Initial state $|001\rangle$
2. $H^{\otimes 3}|001\rangle = \frac{1}{(\sqrt{2})^3}\big(|00\rangle + |01\rangle + |10\rangle + |11\rangle\big) \otimes \big(|0\rangle - |1\rangle\big)$
3. After the oracle

$$\begin{aligned}\frac{1}{(\sqrt{2})^3}\big\{&|00\rangle \otimes (|f(00)\rangle - |\overline{f(00)}\rangle) + |01\rangle \otimes \big(|f(01)\rangle - |\overline{f(01)}\rangle\big)\\ &+ |10\rangle \otimes \big(|f(10)\rangle - |\overline{f(10)}\rangle\big) + |11\rangle \otimes \big(|f(11)\rangle - |\overline{f(11)}\rangle\big)\big\}\end{aligned}$$

Because $f(00) = 1$ and $f(01) = f(10) = f(11) = 0$ we find

$$\begin{aligned}&\frac{1}{(\sqrt{2})^3}\big\{|00\rangle \otimes (|1\rangle - |0\rangle) + |01\rangle \otimes \big(|0\rangle - |1\rangle\big)\\ &\qquad + |10\rangle \otimes \big(|0\rangle - |1\rangle\big) + |11\rangle \otimes \big(|0\rangle - |1\rangle\big)\big\}\\ &= \frac{1}{(\sqrt{2})^3}\big\{-|00\rangle + |01\rangle + |10\rangle + |11\rangle\big\} \otimes (|0\rangle - |1\rangle)\end{aligned}$$

Note that the solution $|00\rangle$ is marked here with a phase -1. This is sometimes called the "kickback phase" phenomenon (like in Deutsch-Josza's algorithm). Now we apply $H^{\otimes 2}$ to the first register and this gives:

$$\frac{1}{(\sqrt{2})^5}\Big\{-|00\rangle - |01\rangle - |10\rangle - |11\rangle + |00\rangle - |01\rangle + |10\rangle - |11\rangle \\ + |00\rangle + |01\rangle - |10\rangle - |11\rangle + |00\rangle - |01\rangle - |10\rangle + |11\rangle\Big\} \otimes (|0\rangle - |1\rangle).$$

We apply the controlled sign change: only $|00\rangle$ changes sign:

$$\frac{1}{(\sqrt{2})^5}\Big\{+|00\rangle - |01\rangle - |10\rangle - |11\rangle - |00\rangle - |01\rangle + |10\rangle - |11\rangle \\ - |00\rangle + |01\rangle - |10\rangle - |11\rangle - |00\rangle - |01\rangle - |10\rangle + |11\rangle\Big\} \otimes (|0\rangle - |1\rangle).$$

Before proceeding, we simplify:

$$\begin{aligned} &\frac{1}{(\sqrt{2})^5}\Big\{-2|00\rangle - 2|01\rangle - 2|10\rangle - 2|11\rangle\Big\} \otimes (|0\rangle - |1\rangle) \\ &\quad = -\frac{1}{(\sqrt{2})^3}\Big\{+|00\rangle + |01\rangle + |10\rangle + |11\rangle\Big\} \otimes (|0\rangle - |1\rangle) \\ &\quad = -\frac{1}{\sqrt{2}}\underbrace{(H^{\otimes 2}|00\rangle)}_{\text{Ô surprise!}} \otimes (|0\rangle - |1\rangle) = -H^{\otimes 3}(|001\rangle). \end{aligned}$$

4. Now we apply the last series of Hadamard gates $H^{\otimes 3}$. Since $H^2 = 1$ we find the final state $-|00\rangle \otimes |1\rangle$. The measurement of the first register gives $x_0 = 00$ with probability 1.