Exercise Set 7: Solution
Quantum Computation

Exercise 1 *Quantum Fourier Transform*

(a) When $M = 2$,

$$QFT = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = H$$

is simply the Hadamard transform.

(b) When $M = 4$,

$$QFT = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \quad \text{so} \quad QFT^\dagger = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

and one can check indeed that $QFT \cdot QFT^\dagger = I$.

(c) By definition, it holds that

$$QFT\,|x\rangle = \frac{1}{2}\left(|0\rangle + i^x\,|1\rangle + (-1)^x\,|2\rangle + (-i)^x\,|3\rangle\right)$$

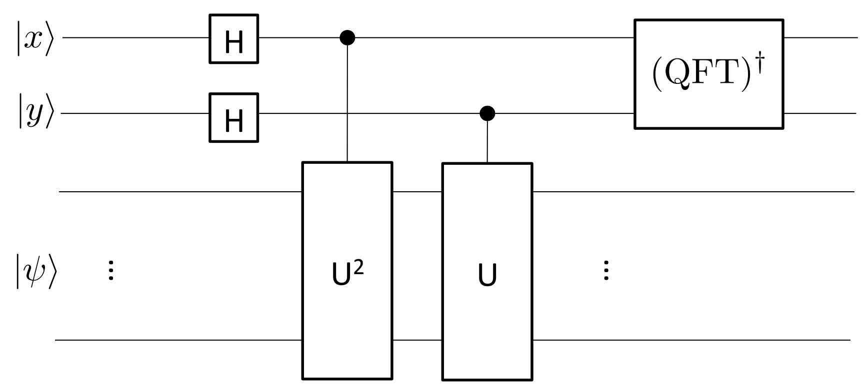
which can be rewritten as

$$QFT\,|x\rangle = \frac{1}{2}\left(|00\rangle + i^x\,|01\rangle + (-1)^x\,|10\rangle + (-i)^x\,|11\rangle\right) = \frac{1}{2}\left(|0\rangle + (-1)^x\,|1\rangle\right)\otimes\left(|0\rangle + i^x\,|1\rangle\right)$$

(d) Even though one may be tempted to deduce from the last expression that QFT can be written as a tensor product, this is not the case! The reason is that x here is a number between 0 and 3 and not a single bit. Formally, one can check by contradiction that there exist no 2×2 matrices A and B such that $QFT = A \otimes B$: the elements of the first row and column of QFT are all equal: this implies that both $a_{11} = a_{12} = a_{21}$ and $b_{11} = b_{12} = b_{21}$; then it becomes impossible to recover $QFT = A \otimes B$.

Exercise 2 *Phase estimation based on the Quantum Fourier Transform*

(a) $\dim(S) = \dim(|x\rangle) \times \dim(|y\rangle) \times \dim(|\psi\rangle) = \dim(\mathbb{C}^2) \times \dim(\mathbb{C}^2) \times \dim\big((\mathbb{C}^2)^{\otimes n}\big) = 2^{2+n}$.
The circuit corresponding to S:



(b) The state after the H's: $H\,|0\rangle \otimes H\,|0\rangle \otimes |u\rangle = \frac{1}{2}\big(\,|00u\rangle + |01u\rangle + |10u\rangle + |11u\rangle\,\big)$
The state after U^{2x} (or R_1): $\frac{1}{2}\big(\,|00u\rangle + |01u\rangle + e^{4\pi i\varphi}\,|10u\rangle + e^{4\pi i\varphi}\,|11u\rangle\,\big)$
The state after U^{y} (or R_2): $\frac{1}{2}\big(\,|00u\rangle + e^{2\pi i\varphi}\,|01u\rangle + e^{4\pi i\varphi}\,|10u\rangle + e^{6\pi i\varphi}\,|11u\rangle\,\big)$

(c) We can write the last expression as

$$\frac{1}{2}\sum_{y_1,y_0\in\{0,1\}} e^{2\pi i\varphi(2y_1+y_0)}\,|y_1,y_0\rangle \otimes |u\rangle = \frac{1}{2}\sum_{y_1,y_0\in\{0,1\}} e^{\frac{2\pi i}{4}(2\varphi_1+\varphi_0)(2y_1+y_0)}\,|y_1,y_0\rangle \otimes |u\rangle$$
$$= \mathrm{QFT}\,|\varphi_1,\varphi_0\rangle \otimes |u\rangle$$

QFT is unitary and therefore $(QFT)^\dagger(QFT) = I_n$. Then, the output state of the circuit is $|\varphi_1\rangle \otimes |\varphi_0\rangle \otimes |u\rangle$.

(d) It suffices to measure the two first qubits because they are $|\varphi_1\rangle \otimes |\varphi_0\rangle$.

Exercise 3 *Effect of imperfections in some gates in Shor's algorithm*

(a) After the Hadamard gates, the state is

$$\begin{aligned}
&\widetilde{H}_0 \otimes \widetilde{H}_1 \otimes \mathbb{I} \otimes \mathbb{I}(|0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle) \\
&= \left(\frac{1}{\sqrt{2}}\right)^2 (|0\rangle + e^{i\varphi_0}|1\rangle) \otimes (|0\rangle + e^{i\varphi_1}|1\rangle) \otimes |0\rangle \otimes |0\rangle \\
&= \frac{1}{\sqrt{4}}(|00\rangle + e^{i\varphi_0}|10\rangle + e^{i\varphi_1}|01\rangle + e^{i(\varphi_0+\varphi_1)}|11\rangle) \otimes |00\rangle \\
&= \frac{1}{\sqrt{4}}(|0\rangle + e^{i\varphi_1}|1\rangle + e^{i\varphi_0}|2\rangle + e^{i(\varphi_0+\varphi_1)}|3\rangle) \otimes |0\rangle
\end{aligned}$$

(b) After the oracle U_f, we obtain the state

$$\frac{1}{\sqrt{4}}(|0\rangle \otimes |f(0)\rangle + e^{i\varphi_1}|1\rangle \otimes |f(1)\rangle + e^{i\varphi_0}|2\rangle \otimes |f(2)\rangle + e^{i(\varphi_0+\varphi_1)}|3\rangle \otimes |f(3)\rangle)$$

Since $f(x) = f(x+2)$, we have:

$$\frac{1}{\sqrt{4}}(|0\rangle + e^{i\varphi_0}|2\rangle) \otimes |f(0)\rangle + \frac{1}{\sqrt{4}}(e^{i\varphi_1}|1\rangle + e^{i(\varphi_0+\varphi_1)}|3\rangle) \otimes |f(1)\rangle$$

Applying the QFT to each term:

$$\frac{1}{4}\sum_{y=0}^{3}(1 + e^{i(\varphi_0+\frac{\pi}{2}2y)})|y\rangle \otimes |f(0)\rangle + \frac{1}{4}\sum_{y=0}^{3}(e^{i(\varphi_1+\frac{\pi}{2}y)} + e^{i(\varphi_0+\varphi_1+\frac{\pi}{2}3y)})|y\rangle \otimes |f(1)\rangle$$

(c) The state right after the measurement is

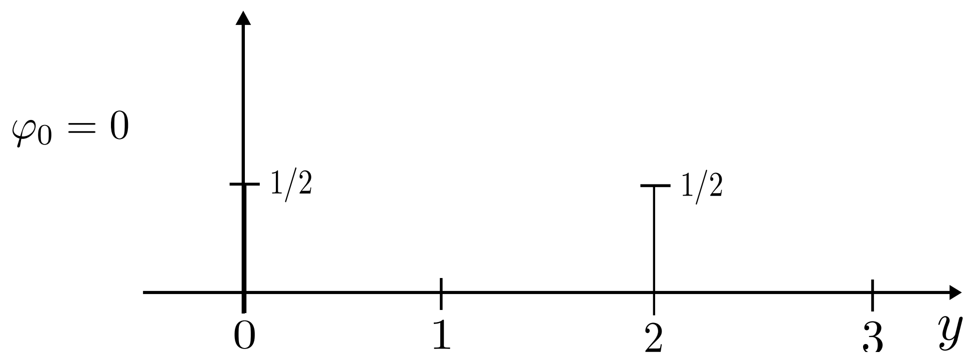
$$|\psi_4\rangle = \frac{1}{4}(1 + e^{i(\varphi_0+\pi y)})|y\rangle \otimes |f(0)\rangle + \frac{1}{4}e^{i(\varphi_1+\frac{\pi}{2}y)}(1 + e^{i(\varphi_0+\pi y)})|y\rangle \otimes |f(1)\rangle.$$

The probability of obtaining y is then given by

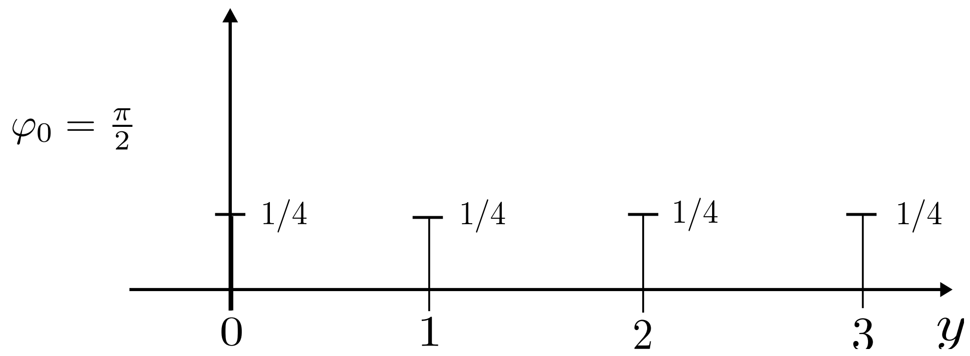
$$\begin{aligned}\text{Prob}(y|\varphi_0,\varphi_1) &= \frac{1}{16}\left\{|1 + e^{i(\varphi_0+\pi y)}|^2 + |1 + e^{i(\varphi_0+\pi y)}|^2\right\} \\ &= \frac{1}{8}\left((1+\cos(\varphi_0+\pi y))^2 + \sin^2(\varphi_0+\pi y)\right)\end{aligned}$$

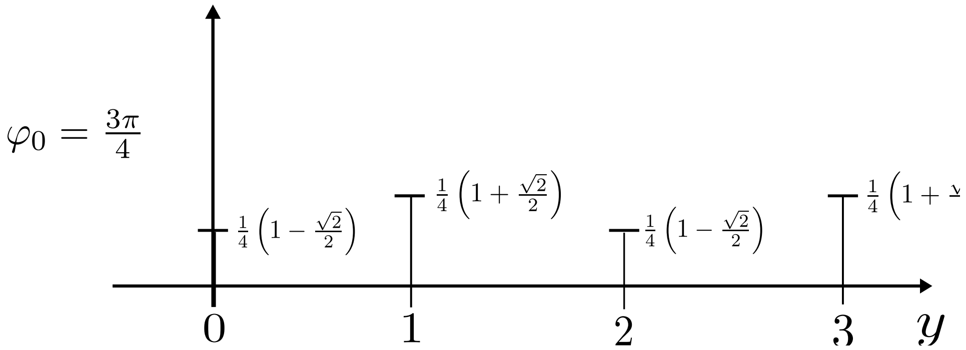
$$\Rightarrow \quad \text{Prob}(y|\varphi_0,\varphi_1) = \frac{1}{4}\left(1 + \cos(\varphi_0+\pi y)\right)$$

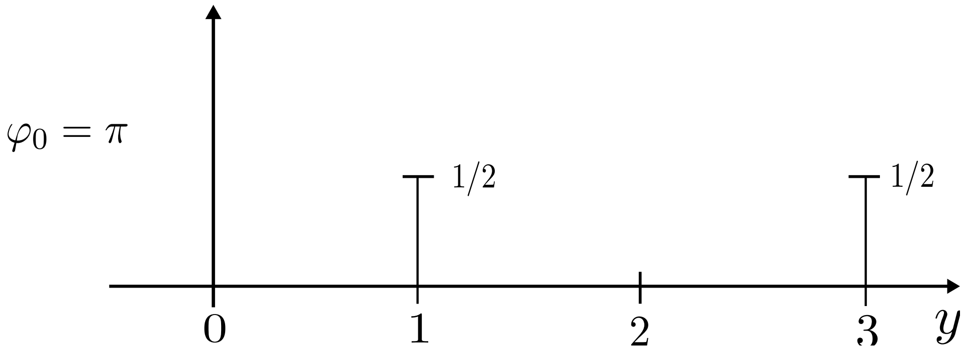
We see that, curiously, this probability does not depend on φ_1. Therefore, Shor's algorithm appears robust to this phase shift.



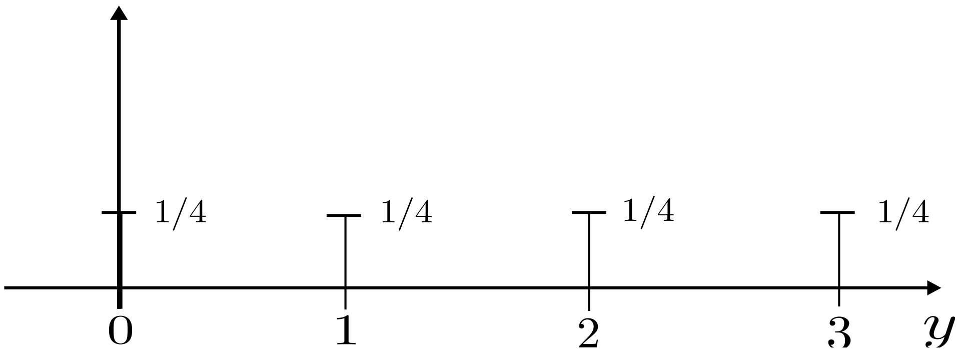
(d)







$$\mathrm{Prob}(y) = \int \mathrm{d}\varphi_0 \, \mathrm{Prob}(y|\varphi_0)\, \mathrm{Prob}(\varphi_0) = \int_0^{2\pi} \frac{\mathrm{d}\varphi_0}{2\pi} \, \mathrm{Prob}(y|\varphi_0) = \frac{1}{4}$$



In an NMR experiment, these spectra are obtained. In the cases when $\varphi_0 = 0, \frac{\pi}{4}, \frac{3\pi}{4}$ or π, we can read the period.