Astrophysics IV, Dr. Yves Revaz — EPFL

4th year physics — Exercises week 5

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Astrophysics IV: Stellar and galactic dynamics

Solutions

Problem 1:

We set up our coordinates such that the slab lays on the $z = 0$ plane. As the mass distribution is discontinuous, we cannot easily rely on the Poisson equation to derive the corresponding potential. We instead use Gauss's law:

$$\int_S \vec{\nabla}\Phi \cdot \mathrm{d}\vec{S} = 4\pi G\, M_S, \tag{1}$$

where S is any surface and M_S is the mass enclosed by the surface S. Let us define S to be the surface of a cylinder perpendicular to the plane $z = 0$. By symmetry (the surface density of the plane is constant) :

$$\vec{\nabla}\Phi = \frac{\partial}{\partial z}\Phi(z) \cdot \vec{e}_z \qquad \text{and} \qquad \frac{\partial}{\partial z}\Phi(z) = -\frac{\partial}{\partial z}\Phi(-z). \tag{2}$$

Thus, in the integral (1) the surface perpendicular to the plane $z = 0$ does not contribute and we get :

$$\int_S \vec{\nabla}\Phi \cdot \mathrm{d}\vec{S} = 2\frac{\partial}{\partial z}\Phi(z)\,\Delta s. \tag{3}$$

where Δs is the surface of the cylinder parallel to the plane $z = 0$. The mass enclosed in the cylinder is :

$$M_S = \Delta s\, \Sigma_0 \tag{4}$$

and (3) with (4) and (1) give :

$$2\frac{\partial}{\partial z}\Phi(z)\,\Delta s = 4\pi G\,\Delta s\,\Sigma_0. \tag{5}$$

This leads to :

$$\frac{\partial}{\partial z}\Phi(z) = 2\pi G\,\Sigma_0, \tag{6}$$

and after integration :

$$\Phi(z) = 2\pi G\,\Sigma_0\, z + \text{const}. \tag{7}$$

Problem 2:

We consider a wire aligned with the x axis. As the mass distribution is discontinuous, we cannot rely on the Poisson equation to derive the corresponding potential. We instead rely on the Gauss Theorem :

$$\int_S \vec{\nabla}\Phi \cdot \mathrm{d}\vec{S} = 4\pi G\, M_S, \tag{8}$$

where S is any surface and M_S is the mass enclosed by the surface S. Let us define S to be the surface of a cylinder of length Δx and radius R, with its symmetry axis being the axis x, i.e., the wire. The surface Δs parallel to the x axis is:

$$\Delta s = 2\pi R\,\Delta x, \tag{9}$$

and the enclosed mass is :

$$M_S = \lambda_0\,\Delta x. \tag{10}$$

By symmetry (the linear density of the wire is constant) :

$$\vec{\nabla}\Phi = \frac{\partial}{\partial R}\Phi(R)\,\vec{e}_R, \tag{11}$$

where $\vec{e}_R$ is perpendicular to the axis x. With (9), (10) and (11), the Gauss theorem becomes :

$$\int_S \vec{\nabla}\Phi \cdot \mathrm{d}\vec{S} = 2\pi R\,\Delta x\,\frac{\partial}{\partial R}\Phi(R) = 4\pi G\,\lambda_0\,\Delta x, \tag{12}$$

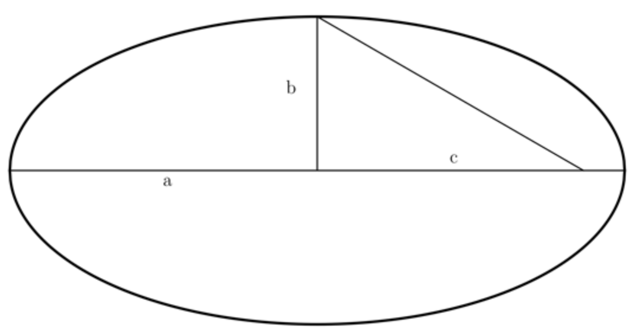
which leads to :

$$\frac{\partial}{\partial R}\Phi(R) = 2\,G\,\frac{\lambda_0}{R}, \tag{13}$$

and after integrating over the radius R :

$$\Phi(R) = 2G\,\lambda_0\,\ln(R) + \mathrm{const}, \tag{14}$$

<u>Problem 3</u>:



The ellipse equation is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{15}$$

the focii are at

$$c = \pm\sqrt{a^2 - b^2}$$

and the eccentricity is defined as

$$e = \frac{c}{a}$$

Using these relations, we write

$$
\begin{aligned}
e^2 &= \frac{c^2}{a^2} = \frac{a^2 - b^2}{a^2} = 1 - \frac{b^2}{a^2} \\
y^2 &= b^2 - \frac{b^2}{a^2} x^2 = \frac{b^2}{a^2}(a^2 - x^2) = (1 - e^2)(a^2 - x^2)
\end{aligned}
$$

We apply a coordinate transformation now: Let $x = x' + ae \ (= x' + c)$. This gives

$$y^2 = (1 - e^2)\left(a^2 - (x' + ae)^2\right) \tag{16}$$

Now we show that the equation of Keplerian orbits (17) can be written in the same form as (16). The Keplerian orbits are defined as

$$r(\varphi) = \frac{a(1 - e^2)}{1 + e\cos(\varphi)} \tag{17}$$

with $x' = r\cos(\varphi)$, $y = r\sin(\varphi)$

$$
\begin{aligned}
r(1 + e\cos(\varphi)) &= r + er\cos(\varphi) = r + ex' \\
&= a(1 - e^2) \\
r^2 &= a^2(1 - e^2)^2 + e^2 x'^2 - 2a(1 - e^2)ex' \\
&= x'^2 + y^2 \\
y^2 &= a^2(1 - e^2) + x'^2(e^2 - 1) - 2a(1 - e^2)ex' \\
&= (1 - e^2)[a^2(1 - e^2) - x'^2 - 2aex'] \\
&= (1 - e^2)[a^2 - a^2e^2 - (x' + ae)^2 + a^2e^2] \\
&= (1 - e^2)[a^2 - (x' + ae)^2]
\end{aligned}
$$

which is exactly equation (16) again.

Problem 4:

First law : The orbit of a planet is an ellipse with the Sun at one of the two foci. This was shown in question 1.

Second law : A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time. Consider the Sun to be at the centre of the coordinate system and a planet at the position $\vec{x}(t)$ with a velocity $\vec{v}(t)$. Consider first the areas sweeps out during an infinitesimal time dt. This area will be:

$$\delta A = \frac{1}{2}\left|\vec{x}(t) \times \mathrm{d}\vec{x}(t)\right|, \tag{18}$$

where $\mathrm{d}\vec{x} = \vec{v}\mathrm{d}t$. So,

$$\delta A = \frac{1}{2}\,\mathrm{d}t\left|\vec{x}(t) \times \vec{v}(t)\right| = \frac{1}{2}\,\mathrm{d}t|\vec{L}|, \tag{19}$$

with $\vec{L}$, the angular momentum (consider a body of unit mass). As the latter is conserved in a spherical potential, δA is independent of the time and of the position along the orbit. We can thus write for any interval time ΔT such that $\Delta T = t_2 - t_1$:

$$A = \int_{t_1}^{t_2} \delta A = \frac{1}{2}\,|\vec{L}| \int_{t_1}^{t_2} \mathrm{d}t = \frac{1}{2}\,|\vec{L}|\,\Delta T, \tag{20}$$

which demonstrates the law.

Third law : The square of a planet's orbital period is proportional to the cube of the length of the semi-major axis of its orbit. From the previous law, we got a result of the form

$$A = \frac{1}{2} L \Delta T,$$

with L the magnitude of the angular momentum of a test particle of unit mass. For a full orbit, $\Delta T \equiv T$ is the period, and A is the area of the ellipse:

$$A = \pi ab = \pi a^2 \sqrt{1 - e^2}.$$

Let us now turn our attention to L. There are different ways of calculating it, but we will use the Vis-Viva equation:

$$v^2(r) = GM\left(\frac{2}{r} - \frac{1}{a}\right).$$

Let's take , e.g., $r = r_{\min}$:

$$v^2(r_{\min}) = GM\left(\frac{2}{r_{\min}} - \frac{1}{a}\right) = GM\left(\frac{2a - r_{\min}}{r_{\min} a}\right)$$

but $2a - r_{\min}$ is $r_{\max}$, and we also have $r_{\min} r_{\max} = b^2$. Together we get:

$$v^2(r_{\min}) = \frac{GM}{a}\left(\frac{b}{r_{\min}}\right)^2$$

So we have

$$L = L(r_{\min}) = \sqrt{\frac{GM}{a}} b = \sqrt{\frac{GM}{a}} a \sqrt{1 - e^2}$$

Thus the period is

$$T = \frac{2A}{L} = 2\frac{\pi a^2 \sqrt{1 - e^2}}{\sqrt{\frac{GM}{a}} a \sqrt{1 - e^2}} = 2\frac{\pi a^{3/2}}{\sqrt{GM}},$$

or

$$T^2 = \frac{4\pi^2}{GM} a^3.$$

Throughout this exercise, we took a test particle of unit mass to make dealing with the units easier. (Usually, $L = mrv$ and not only $L = rv$ which we used here.)