MULTIPLE-VARIABLE ANALYSIS OF FACTORS AFFECTING LIGHTNESS AND SATURATION

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Investigations of hue, lightness, and saturation have been made usually by means of small patches of spectrally homogeneous light in such fashion that the influence of only one factor at a time was studied. When experimental procedures are properly designed it is possible to study the effect of several factors operating jointly and to isolate each statistically by means of methods generally known as analysis of variance. In the present study the relative importance of composition and amount of illumination and reflectance of background on lightness and saturation of reflecting surfaces will be evaluated through the use of analysis of variance.[1] The application of this statistical method to a problem in color vision involving triple classification of data into groups (amount of illumination), columns (composition of illumination), and rows (reflectance of background), will be illustrated and explained.

The data discussed here were obtained under conditions which differ in important respects from the usual approaches: instead of small, foveal stimuli, the whole retina was subjected to chromatic light and several variables were allowed to operate simultaneously during the observations in which the *Ss* estimated lightness and saturation by comparing all stimuli within the field but without equating or matching in the usual manner. The method of visual estimation employed in this work is open to objections of which the writer is aware but it is justified by the fact it is the

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[1] Although a number of studies have appeared involving use of analysis of variance and factorial design in psychological problems none has yet reached the writer's attention where the application is to a purely perceptual problem. The following works are of interest for the problems discussed here: R. S. Crutchfield, Efficient factorial design and analysis of variance illustrated in psychological experimentation, *J. Psychol.,* 5, 1938, 339-346; The determiners of energy expenditure in string-pulling by the rat, *ibid.,* 7, 1939, 163-178; R. S. Crutchfield and E. C. Tolman, Multiple-variable design for experiments involving interaction of behavior, *Psychol. Rev.,* 47, 1940, 38-42; R. B. Hackman, An experimental study of variability in ocular latency, *J. Exper. Psychol.,* 27, 1940, 546-558; Brent Baxter, The application of factorial design to a psychological problem, *Psychol. Rev.,* 47, 1940, 494-500; Problems in the planning of psychological experiments, this JOURNAL, 54, 1941, 270-280. For an example of triple classification similar to the one employed for the data discussed here see P. R. Rider, *An Introduction to Modern Statistical Methods,* 1939, 142-150. I am indebted to Dr. R. S. Crutchfield and Dr. D. B. Judd for several helpful criticisms and suggestions in the preparation of this article.

only method possible when the whole retina is being stimulated.[2] The consistency of results from 5 *Ss* who judged 18 Munsell samples a total of 48 times each and the fact that reliable differences have been found

TABLE I

AVERAGE SATURATION-ESTIMATES FOR 18 SAMPLES VIEWED ON EACH OF THREE BACKGROUNDS UNDER ILLUMINATIONS OF FOUR DIFFERENT SPECTRAL COMPOSITIONS, EACH ILLUMINATION BEING USED IN FOUR DIFFERENT AMOUNTS

Foot candles	Background	Hues R	Y	G	B	Total	Av.
	W	3.33	4.64	3.14	2.36	13.47	
135	G	3.50	4.69	4.08	3.69	15.96	
	Bk	4.77	4.75	4.58	4.42	18.52	
	Total	11.60	14.08	11.80	10.47	47.95	3.99
	W	2.86	4.14	3.61	3.25	13.86	
72	G	4.20	5.14	4.58	4.17	18.09	
	Bk	5.97	5.00	4.11	4.94	20.02	
	Total	13.03	14.28	12.30	12.36	51.97	4.33
	W	2.32	3.82	3.83	2.47	12.44	
4.5	G	3.99	4.22	3.44	3.69	15.34	
	Bk	5.05	4.44	3.65	4.05	17.19	
	Total	11.36	12.48	10.92	10.21	44.97	3.75
	W	2.86	4.22	3.55	2.18	12.81	
1.4	G	3.42	4.28	4.32	2.93	14.95	
	Bk	4.55	4.19	3.35	3.11	15.20	
	Total	10.83	12.69	11.22	8.22	42.96	3.58
	W	11.37	16.82	14.13	10.26	52.58	3.28
All	G	15.11	18.33	16.42	14.48	64.34	4.02
	Bk	20.34	18.38	15.69	16.52	70.93	4.43
	Grand total	46.82	53.53	46.24	41.26	187.85	
	Av.	3.90	4.46	3.85	3.44		

statistically for the different conditions of observation justify this method of assessing lightness and saturation.

PROCEDURE

The experimental conditions under which data in Tables I and IV were obtained have been fully described elsewhere,[3] so only briefest mention of them will be

[2] For discussion of the method of visual estimation of color attributes and references to earlier uses of the method, see S. M. Newhall, Preliminary report of the O.S.A. subcommittee on the spacing of the Munsell colors, *J. Opt. Soc. Amer.*, 30, 1940, 617-645.

[3] Harry Helson, Fundamental problems in color vision: I. The principle governing changes in hue, saturation, and lightness of non-selective samples in chromatic illumination, *J. Exper. Psychol.*, 23, 1938, 439-476; and Harry Helson and V. B. Jeffers, Fundamental problems in color vision: II. Hue, lightness, and saturation of selective samples in chromatic illumination, *ibid.*, 26, 1940, 1-27. The data discussed in the present article were obtained with the samples used in the second of these studies and consisted of 15 selective Munsell papers and three non-selective cardboard samples identical with the backgrounds.

made here to enable the reader to interpret the results. Estimations of lightness and saturation were made by the *S*s on a 11-point scale running from 0-10 for each attribute. The use of a scale with definite end-points tends to restrict discriminations and operates against the discovery of differences due to different conditions yet in spite of this fact the effects of the variables being studied appear very clearly. The values given in Tables I and IV are averages of 90 observations for each condition being discussed.

Variants: Illuminants, illuminance, and background. Eighteen selective and non-selective samples ranging from 0.03 to 0.80 in reflectance for each illuminant were observed under four lamp-filter combinations having yellowish-red, reddish-yellow, yellowish-green, and reddish-blue hues. They will be denoted by their dominant component as R, Y, G, and B illuminants respectively. According to their locations in the trilinear mixture diagram the R and Y illuminants are practically equivalent to spectrally homogeneous lights while the G and B illuminants must be regarded as having a fair admixture of 'white' light.[4] Three backgrounds were used of daylight white, gray, and black cardboard having reflectances of 0.80, 0.23, and 0.03 for each of the four illuminants. Four amounts of illumination (hereafter referred to as "illuminance" to indicate intensity of illumination in accordance with modern usage) were employed with each filter as given in Tables I and IV. Due to differences in transmission of the filters these values, 135, 72, 4.5, and 1.4 f.c. must be multiplied by the coefficients of transmission to obtain the actual illuminance on the samples for each level of illuminance. The transmissions are: R, 0.052; Y, 0.362; G, 0.038; and B, 0.006. It is seen that while the absolute illuminances differed from filter to filter the *ratios* were the same. The second highest illuminance for each filter is approximately that required to yield maximum saturation for a light color on a dark ground.

Results

Saturation. Data for saturation are given in Table I which is arranged to show the three-fold reference of each value in the table and to facilitate calculations of sums of squares. Average saturations found under each condition, illuminance, composition of illuminant, and background, appear at the bottom and side of the table. From these data we shall attempt to answer such questions as these. Does each condition being studied affect saturation operating singly? Do these conditions operating jointly affect saturation? The statistics on which answers to these and other questions depend are easily calculated once the data are properly arranged. From the sum of squares for any variable or condition the mean square or variance is obtained by dividing by the appropriate degrees of freedom (Table II). Significance is determined by taking the ratio of each variance to a variance chosen as the 'error variance.' This variance is due to uncontrolled, chance, or randomized factors in the experiments and depends upon the experimental design. In the experimental design employed here the triple interaction variance is usually taken as the error variance against which all other variances are tested for significance. The ratio of any variance

[4] The trilinear coördinates of the illuminants and of the samples in each illuminant as well as plots of the illuminant-sample combinations will be found in D. B. Judd, Hue, saturation and lightness of surface colors with chromatic illumination, *J. Opt. Soc. Amer.*, 30, 1940, 2-32.

to the error variance is called "F." The significance of F depends upon the degrees of freedom in the variances entering into the ratio. Tables giving F-values for various degrees of freedom at so-called 5% and 1% levels of significance are readily obtainable.[5] F-values larger than the value at the 1% level or point are highly significant since the chance of occurrence of an F-value as large as that at the 1% point is only one in 100. F-values between the 1% and 5% points may be significant while F-values smaller than those at the 5% point are not considered significant since they may occur by random selection as often as 5 or more times in a hundred.

Comparison of the F-values in column 4 of Table II with the F-values of column 5 shows that composition of illuminant has a highly significant effect on saturation ($F = 22.1$ as against 5.09 at the 1% point). Illuminance and especially reflectance of background also have highly significant effects on saturation. The obtained F-values are from 2.5 to 9.0 times the 1% values for F and are all highly significant. Background effects are most significant since the F-value in question is the largest.[6] But since we have four illuminances, four different compositions of illuminants, and

TABLE II

ANALYSIS OF VARIANCE SUMMARY FOR SATURATION

Variants	Sums of squares	Degrees of freedom	Mean square	F	5% and 1% pts.	
(1) Illuminants	6.35	(4−1)= 3	2.12	22.1	3.16	5.09
(2) Illuminances	3.83	(4−1)= 3	1.28	13.3	3.16	5.09
(3) Backgrounds	10.18	(3−1)= 2	5.09	53.0	3.55	6.01
(1)×(2)	1.16	3×3= 9	0.13	1.3	2.51	3.71
(1)×(3)	6.15	3×2= 6	1.02	10.6	2.66	4.01
(2)×(3)	1.71	3×2= 6	0.29	3.0	2.66	4.01
(1)×(2)×(3)	1.73	3×3×2=18	0.096			
Total	31.11	(4×4×3)−1=47				

three reflectances of background, the question arises are all variations in any given condition significant? The F-test obviously applies to the data as a whole and does not tell which differences between averages are significant. The value of the F-test lies in the fact that we do not need to test for specific differences if it does not yield a significant result.[7]

Before proceeding with tests of differences within each variable let us complete our inspection of the F-values in Table II. The remaining F-values refer to interaction effects between the variables and significant F-values mean that there is inter-

[5] For a single table containing values of F and t see G. W. Snedecor, *Calculation and Interpretation of Analysis of Variance and Covariance,* 1934, 88-91.

[6] Newhall, *op. cit.,* failed to find systematic differential effects of background on hue and saturation, but it must be remembered that he worked in fairly white illumination where background effects, except on lightness, which he found, are minimal. Simultaneous comparison of samples on backgrounds differing greatly in reflectance reveals differences in hue and saturation even in white illuminance. Newhall's subjects did not judge samples simultaneously on different backgrounds. In strongly chromatic illuminants, background effects on saturation are so great they appear even in the reports made on each background in different observational periods.

[7] For discussion of this point consult E. F. Lindquist, *Statistical Analysis in Educational Research,* 1940, 95 ff.

action in the operation of the conditions being studied. It is seen that only the interaction between illuminants and backgrounds is highly significant with an F-value of 10.6 against a theoretical value of 4.01 at the 1% point. This means there is interaction between composition of illuminant and reflectance of background so that saturation is affected by the interaction of these variables as well as directly and hence the effect of either cannot be studied without taking into consideration the other. Previously published results and interpretations from other points of view bear out this conclusion:[8] backgrounds of low reflectance induce the hue of the illuminant whereas those of high reflectance induce the after-image complementary hue with important effects on saturation as a result. The F-test makes possible statistical test and expression of this interaction which was known from observation and the principle of color conversion.

Interaction between reflectance of background and illuminance may be significant since its F-value lies between the 1% and 5% points. Some interaction between these variables might have been expected because *lower* reflectances have the tendency

TABLE III

Differences between Averages in Saturation and Minimum Differences Significant at the 5% and 1% Levels

Illuminants	Backgrounds	Illuminances
Y—R: 4.46—3.90=0.56	B—W: 4.43—3.28=1.15	I—II: 3.99—4.33=—0.34
Y—G: 4.46—3.85=0.61	B—G: 4.43—4.02=0.41	I—III: 3.99—3.75= 0.24
Y—B: 4.46—3.44=1.02	G—W: 4.02—3.28=0.74	I—IV: 3.99—3.58= 0.41
R—G: 3.90—3.85=0.05		II—III: 4.33—3.75= 0.58
R—B: 3.90—3.44=0.46		II—IV: 4.33—3.58= 0.75
G—B: 3.85—3.44=0.41		III—IV: 3.75—3.58= 0.17
5% point=0.265	5% point=0.229	5% point=0.265
1% point=0.363	1% point=0.314	1% point=0.363

to induce illuminant hue just as *higher* illuminances do, but owing to the extremely large variations in illuminance necessary for clear-cut effects on colors further data are needed to substantiate this finding. In some cases F-values between the 1% and 5% points may be regarded as significant but with the knowledge that general illuminance is of secondary importance we should prefer to reach the 1% level of significance before claiming interaction of this variable with any others.

The F-value for interaction of composition of illuminant and illuminance is too small to possess any significance. Had equal amounts of illumination been used for all illuminants at each level of illuminance significant interaction effects may have been established between these variables and saturation since it is known that the various regions (hues) of the spectrum reach maximum saturation at different levels of intensity.

Having discussed the results of the F-test for each of the variables and their interaction we may now turn to the individual differences between averages for each condition and employ the "t-test" to determine which are significant. The use of the t-test in place of the classical standard error of the difference simplifies calculations, makes possible the use of a better estimate of the variance of the

[8] Helson, and Helson and Jeffers, *opp. citt.*

means being tested, and enables us to determine the difference necessary for significance between any two means at the 1% or 5% levels.[9]

From Table III it is seen that the differences in saturation between all illuminants, except R and G are highly significant since the differences are larger than 0.363, the value at the 1% point for the degrees of freedom involved. The average saturation is highest for the Y illuminant, next highest for the R and G which do not differ significantly, and lowest for the B illuminant. Since the Y and R illuminants plot nearer the spectral locus in the color mixture diagram than the G and B this result is to be expected and is further confirmation of the validity of the method of visual estimation of color attributes with trained *S*s and sufficient observations.

That reflectance of background also exerts important effects on saturation is shown by the fact that all differences between backrounds are larger than the value, 0.314, at the 1% point. The highest saturation is found on the black background, the lowest on the white, with an intermediate value on the gray. Due to the fact that even in homogeneous illumination more than a single hue as well as achromatic colors can be seen, in accordance with the general principle of color conversion, the average saturations discussed here represent net effects of the viewing conditions on saturation. Light backgrounds, through their induction of the after-image complementary to the illuminant hue tend to have a net desaturating effect when all samples in the field of view are considered as is the case when averages are used.

Of the six differences between averages for saturation as a function of illuminance, three are highly significant and concern the two higher illuminances as against the two lower (cf. Table III). One difference falls between the 1% and 5% points and is therefore doubtful while the other two differences are not large enough to fall even within the 5% point and are hence not significant. The effects of illuminance on saturation cannot be said to be clear-cut unless extremely large differences in illuminance are in question. This is shown by the fact that the least difference in illuminance that is reliably significant is that between 72 and 4.5 f.c. which involves a ratio of 16 to 1. We saw above that the difference between white

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[9] The calculation of the minimum difference significant by the t-test is as follows: $t = M_1 - M_2/\sigma_{diff.}$ But the value of t at the 1% or 5% level depends only on the degrees of freedom entering into t and hence can be found in a table of t. Knowing t we can determine the minimum difference necessary for significance at the 1% or 5% level by using the appropriate value of t in the following formula:

$$\text{Minimum difference significant} = t \times \sigma_{diff.}$$

Calculation of $\sigma_{diff.}$ involves the standard error of the means whose difference we are testing. Since we are using the variance obtained from *all* the data as our estimate of the variance of the distributions whose means we are testing, the standard error of any mean becomes the error variance (in our case 0.096, Table II) divided by the number of cases on which the mean is based. Bearing in mind the classical formula:

$$\sigma_{diff.} = (\sigma^2_{m_1} + \sigma^2_{m_2})^{1/2} = (2 \times 0.096/N)^{1/2}$$

it is seen that the error variance used in the F-test can be used in the t-tests, thereby saving labor in computing as well as providing a better estimate of the variances of the means. The value of N depends upon the number of cases on which any particular mean depends. For testing differences between means for illuminants, N = 12; for backgrounds, N = 16; and for illuminances N = 12, as reference to Tables I and IV will show. In deriving the variance of means we use N and not N-1 because the latter value has already been used in the calculation of the error variance. On the use of the t-test, cf. Lindquist, *op. cit.*, 97.

and gray backgrounds was significant although the ratio of reflectances of the two was only 3.5 to 1, showing how much more effective change in reflectance of background is on saturation than change in general illuminance. The fact that the highest illuminance did not result in highest saturation indicates a falling off in saturation after a certain point has been reached and may be a parallel to the reduction in saturation found with small, spectrally homogeneous stimuli with increasing intensity.

Lightness. The data in Table IV for lightness permit us to evaluate the effects of composition of illuminant, amount of illumination, and reflectance of background

TABLE IV

Average Lightness Estimates for 18 Samples Viewed on Each of Three Backgrounds under Illuminations of Four Different Spectral Compositions, Each Illumination Being Used in Four Different Amounts

Foot candles	Background	Hues R	Y	G	B	Total	Av.
	W	4.33	5.08	4.75	4.47	18.63	
135	G	4.99	5.36	5.25	4.94	20.54	
	Bk	4.77	5.50	4.94	5.11	20.32	
	Total	14.09	15.94	14.94	14.52	59.49	4.96
	W	4.86	5.08	4.92	4.94	19.80	
72	G	5.03	5.14	5.14	5.16	20.47	
	Bk	4.47	5.67	5.03	4.92	20.09	
	Total	14.36	15.89	15.09	15.02	60.36	5.03
	W	4.19	4.86	4.40	4.26	17.71	
4.5	G	4.30	4.93	4.65	4.47	18.35	
	Bk	4.61	5.38	4.66	4.36	19.01	
	Total	13.10	15.17	13.71	13.09	55.07	4.59
	W	4.18	4.87	4.55	4.10	17.70	
1.4	G	3.92	4.96	5.01	4.22	18.11	
	Bk	4.22	5.14	4.62	4.19	18.17	
	Total	12.32	14.97	14.18	12.51	53.98	4.50
	W	17.56	19.89	18.62	17.77	73.84	4.61
All	G	18.24	20.39	20.05	18.79	77.47	4.84
	Bk	18.07	21.69	19.25	18.58	77.59	4.85
	Grand total	53.87	61.97	57.92	55.14	228.90	
	Av.	4.49	5.16	4.84	4.49		

in a manner analogous to the treatment of saturation made above. The statistics involved are similar. Since the obtained values of F are so much larger than the F-values at the 1% point in the case of illuminants, illuminances, and backgrounds we can be very sure that these conditions have important effects on lightness. The average lightness for the four illuminants are: Y, 5.16; G, 4.83; B, 4.60; and R, 4.49. The F-test for illuminants tells us only that the test for all illuminants yields a result hardly attributable to chance. To discover which illuminants yield significantly higher or lower lightness values we must resort to the t-test for each illuminant paired with every other. In Table VI are given the differences in lightness between every pair of illuminants and the minimum differences at the 5% and 1% levels of significance. It is evident that all differences between illuminants are highly significant, being larger than the minimum difference necessary for significance at the 1% point, with the exception of that between R and B. The difference in lightness of 0.10 step between R and B is less than 0.187, the minimum difference at the 5% level. We are therefore justified in concluding that in the Y illuminant lightness is judged as higher than in any of the others. The next highest average lightness is found in the G illuminant which differs significantly from B and R in which the lowest lightness values are found. At the moment no satisfactory explanation can be given for these statistically significant differences in estimations of lightness in the four illuminants.[10]

We see from Table V that background exerts a statistically reliable effect on lightness but from Table VI we find by the t-test that only the differences between

TABLE V

ANALYSIS OF VARIANCE SUMMARY FOR LIGHTNESS

Variants	Sums of squares	Degrees of freedom	Mean square	F	5% and 1% pts.	
(1) Illuminants	3.22	(4−1)= 3	1.07	41.0	3.16	5.09
(2) Illuminances	2.51	(4−1)= 3	0.84	32.0	3.16	5.09
(3) Background	0.57	(3−1)= 2	0.28	11.0	3.55	6.01
(1)×(2)	0.43	3×3= 9	0.05	1.83	2.51	3.71
(1)×(3)	0.33	3×2= 6	0.05	2.12	2.66	4.01
(2)×(3)	0.28	3×2= 6	0.05	1.79	2.66	4.01
(1)×(2)×(3)	0.47	3×3×2=18	0.026			
Total	7.81	(4×4×3)−1=47				

white and gray and white and black backgrounds are statistically reliable. The difference of 0.01 step in average lightness between gray and black backgrounds is too far below the 5% level to have any significance. This is because the stimuli were observed on the different backgrounds at different times so that some background effect was lost as *S*s tend to give the lightest sample the maximum value (10) and the darkest sample the minimum (0) allowed in the scale. Yet in spite of this 'scale-effect' the influence of background is revealed since two out of the three possible differences are significant.

Turning now to the effect of amount of illumination on judgments of lightness we find from Table VI that four out of the six differences are larger than the minimum difference at the 1% level, the other two differences being much smaller than the difference expected at the 5% level. These results of the t-tests point clearly to the fact that the higher illuminances with average lightness values of 4.96 and 5.03 yield significantly higher lightness values than the two lower ones with averages of

[10] It is difficult to account for these results on the basis of photometric brilliance of the samples in the four illuminants. Photometrically the average brilliance of the samples was 8.01 in the yellow, 1.20 in the red, 0.88 in the green, and 0.12 in the blue in the second highest illuminance (72 f.c.). This explains the highest lightness in the yellow but leaves unexplained the other three illuminants if lightness is purely a function of photometric brilliance (background effects being ruled out). Purity or chromaticity of illuminant does not explain because the red was the most homogeneous and the blue the least homogeneous illuminant. There is some basis for supposing the mid-spectral hues to have higher intrinsic brilliance and this may account for the higher lightness ascribed to the samples in the yellow and green illuminants as against the red and the blue.

4.59 and 4.50. The differences in lightness between the two higher illuminances and between the two lower illuminances are not statistically significant, showing that it is necessary to more than triple illuminance in order to obtain significant increase in lightness.

Interaction effects on lightness of the conditions studied here are absent so far as these data go. The F-values for illuminants and illuminances, illuminants and backgrounds, and illuminances and backgrounds are all below the 5% level and hence are not significant. At first sight this may seem surprising in view of the clear-cut effects on lightness which each condition has been found to exert by itself, but consideration of the experimental conditions and certain established facts explain our failure to find interaction. First, composition of illuminant and amount of illuminant affect the lightness of the whole field and as pointed out above it takes

TABLE VI

DIFFERENCES IN LIGHTNESS AND MINIMUM DIFFERENCES STATISTICALLY SIGNIFICANT AT THE 5% AND 1% POINTS

Illuminants		Backgrounds	Illuminances	
Y−R: 5.16−4.49=	0.67	B−W: 4.85−4.61=0.24	I−II: 4.96−5.03=	−0.07
Y−G: 5.16−4.84=	0.32	B−G: 4.85−4.84=0.01	I−III: 4.96−4.59=	0.37
Y−B: 5.16−4.59=	0.57	G−W: 4.84−4.61=0.23	I−IV: 4.96−4.50=	0.46
G−R: 4.84−4.49=	0.35		II−III: 5.03−4.59=	0.44
G−B: 4.84−4.59=	0.25		II−IV: 5.03−4.50=	0.53
R−B: 4.49−4.59=	−0.10		III−IV: 4.59−4.50=	0.09
5% point=0.136		5% point=0.118	5% point=0.136	
1% point=0.187		1% point=0.161	1% point=0.187	

extremely large variations in conditions affecting the whole field to produce significant results in color. Secondly, the use of different absolute illuminances would tend to obscure interaction effects between these two conditions and any other condition simultaneously operative. Finally, the well-known constancy of lightness tends, perhaps more than anything else, to prevent detection of differences unless they are quite large. Practically this means that the eye is an extremely labile mechanism in its adjustments for lightness-vision in complex situations.

NOTE ON COMPUTATIONS FOR ANALYSIS OF VARIANCE

In any analysis of variance, we use sums of squares because sums of squares are additive and allow partition into components and the inverse operation by which sums of squares of components add to total sum of squares. It is first necessary to determine the total sum of squares of deviations of all items entering into the analysis from the general mean. Inspection of Tables II and V reveals that sums of squares for the various components add to "Total" but the mean squares or variances do not add to the total variance. The formula for computing total sum of squares is simple when the data are properly arranged for each case:

$$\text{Total sum of squares} = \Sigma X^2 - T^2/N \ldots\ldots\ldots [1]$$

where X refers to each item in Table I or V (apart from the totals and averages), T is the sum of all items in the table, and N is their number. The T^2/N term in the formula is the correction factor made necessary by the fact that the original values were squared instead of their deviations from the general mean. Although other 'total' sums of squares, hereafter called 'sub-total' sums, will be found, the correction factor to be used will in every case be the same: T^2/N. The computations for the data in Table I follow:

$$\text{Total sum of squares} = (3.33^2+4.64^2+3.14^2+2.36^2+3.50^2 \ldots +3.11^2)-187.85^2/48$$
$$= 766.27-735.16 = 31.11$$

In order to isolate each of the variables and to calculate the sum of squares of the deviations of the means of any given variable from the general mean we abstract the totals for each variable in Table I and form Tables VII, VIII and IX of illuminants and illuminances, illuminants and backgrounds, and illuminances and backgrounds. This procedure not only has the advantage of making it possible to treat a problem in three-fold classification as one in two-fold classification, thereby simplifying matters, but it also shows more clearly where the items come from in each subsequent step of the computations.

TABLE VII

ILLUMINANTS AND ILLUMINANCES

Footcandles	Hues				Total
	R	Y	G	B	
135.0	11.60	14.08	11.80	10.47	47.95
72.0	13.03	14.28	12.30	12.36	51.97
4.5	11.36	12.48	10.92	10.21	44.97
1.4	10.83	12.69	11.22	8.22	42.96
Total	46.82	53.53	46.24	41.26	187.85

TABLE VIII

ILLUMINANTS AND BACKGROUNDS

Background	Hues				Total
	R	Y	G	B	
White	11.37	16.82	14.13	10.26	52.58
Gray	15.11	18.33	16.42	14.48	64.34
Black	20.34	18.38	15.69	16.52	70.93
Total	46.82	53.53	46.24	41.26	187.85

TABLE IX

ILLUMINANCES AND BACKGROUNDS

Backgrounds	Foot candles				Total
	135	72	4.5	1.4	
White	13.47	13.86	12.44	12.81	52.58
Gray	15.96	18.09	15.34	14.95	64.34
Black	18.52	20.02	17.19	15.20	70.93
Total	47.95	51.97	44.97	42.96	187.85

Let us call the total sum of squares for the data of Tables VII, VIII, and IX, sub-total sums to distinguish them from the total sum of Table I. Since each value in these tables is a sum of several items from the original Table I the formula for total sum of squares (Formula 1) must be modified as follows:

$$\text{Sub-total sum} = 1/n(\Sigma X^2_s) - T^2N \quad \ldots\ldots\ldots\ldots [1a]$$

where n is equal to the number of items in Table I entering into each item in the sub-tables and X_s is each item in a sub-table.

For illuminants and intensities we calculate from Table VII according to formula 1a:

$$\text{Sub-total sum of squares} = 1/3(11.60^2 + 14.08^2 + \ldots + 8.22^2) - 735.16 = 11.34$$

For the calculation of the sum of squares of the deviations of the means of any variable from the general mean we use the totals for the variable in question without regard to the action of the other variables since they will be isolated in similar fashion. Hence for the sum of squares of the deviations of the means of rows or columns in Tables VII, VIII, and IX we use the formulae:

$$\text{Sums of squares for rows} = 1/n_{tr}(\Sigma T^2_r) - T^2/N \quad \ldots\ldots\ldots\ldots [2]$$
$$\text{Sums of squares for columns} = 1/n_{tc}(\Sigma T^2_c) - T^2/N \quad \ldots\ldots\ldots\ldots [3]$$

Here T_r and T_c refer to totals of rows and columns respectively and n_{tr} and n_{tc} refer to the number of items entering into each T_r and T_c. It is always best to check the number of items entering into T_r and T_c by referring each total back to the original summary of data (Table I).

The sums of squares for illuminants and illuminances are found as follows by formulae [2] and [3] from the data in Table VII.

$$\text{For illuminants:} \quad 1/12(46.82^2 + 53.53^2 + 46.24^2 + 41.26^2) - 735.16 = 6.35$$
$$\text{For illuminances:} \quad 1/12(47.95^2 + 51.97^2 + 44.97^2 + 42.96^2) - 735.16 = 3.83$$

The interaction sum of squares for illuminants and illuminances can now be found as a remainder when the sums of illuminants and illuminances are subtracted from the subtotal:

$$\text{Interaction illuminants} \times \text{illuminances} = 11.34 - (6.35 + 3.83) = 1.16$$

We have next to calculate the sub-total sum of squares for Table VIII and the sum for backgrounds in this table, the calculation of illuminants being unnecessary because we have just found this value from Table VII to be 6.35. Sub-total sum for illuminants and backgrounds:

$$1/4(11.37^2 + 16.82^2 + \ldots\ldots\ldots\ldots + 16.52^2) - 735.16 = 22.68$$

The sum of squares for background is:

$$1/16(52.58^2 + 64.34^2 + 70.93^2) - 735.16 = 10.18$$

The interaction term for illuminants by backgrounds is again found as a remainder:

$$22.68 - (10.18 + 6.35) = 6.15$$

From Table IX there remains to be calculated only the sub-total sum of squares for illuminances and backgrounds in order to obtain the interaction value for these two factors since we have already found the sum of squares for each of them from the data in Tables VII and VIII. The sub-total sum is found to be:

$$1/4(13.47^2 + 13.86^2 + \ldots\ldots\ldots\ldots + 15.20^2) - 735.16 = 15.72$$

The interaction of illuminances and background is given as a remainder:

$$15.72-(10.18+3.83)=1.71$$

Only one value is left to be computed, that for the interaction of illuminants, illuminances and backgrounds which is the remainder due to so-called 'error' and is used as the term against which all other factors are tested. It is equal to total sum minus the sum of the sums of squares found above (excepting the sub-total sums):

$$31.11-(6.35+3.83+10.18+1.16+6.15+1.71)=1.73$$

We now have our sums of squares and can reduce them to mean squares or variances by dividing each by its appropriate degrees of freedom. Since there are four illuminants the degrees of freedom will be three, similarly, there are three degrees of freedom for illuminances and two for backgrounds. The degrees of freedom for interaction are equal to the product of the degrees of freedom of the factors which are interacting. From Table II it is seen there are nine for illuminants and illuminances, six for illuminants and backgrounds, six for illuminances and backgrounds, and 18 for illuminants, illuminances and backgrounds. The degrees of freedom for all components must add to the total number of degrees of freedom and this furnishes a check on whether we have assigned the proper degrees of freedom to each component of the analysis. There were 48 items in our original Table I which allow a total of 47 degrees of freedom. The component degrees of freedom in Table II add to 47.

Summary

Saturation and lightness are found by analysis of variance to be significantly affected by strongly chromatic illuminants, by great changes in amounts of illumination, and by differences in reflectance of backgrounds. Interaction effects appear between illuminants and backgrounds and possibly between illuminances and backgrounds with respect to saturation but no interaction of these variables has been found for lightness in the experiments reported here. The computations involved in the analysis of variance are shown and explained.