# Potential Theory II

# Outlines

Potential Theory : general results

- Gauss Law
- Poisson Equation
- Total potential energy

Spherical systems:

- Newton's Theorems
- Circular speed, circular velocity, circular frequency, escape speed, potential energy

Examples of spherical models:

- "Potential based" models
- "Density based" models

Axisymmetric models for disk galaxies

- "Potential based" models

## Potential theory : general results

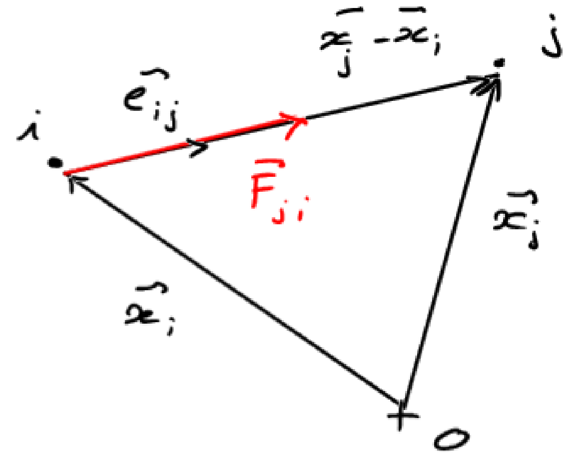
Goal : compute the gravitational potential / forces due to a large number of stars ( point masses )

$N \sim 10^{11}$ for a Milky Way like galaxy

As the relaxation time of such system is very large ( $\gg$ the age of the Universe) we can describe the system with a smooth analytical potential / density .

# Newton Law

$$\vec{F}_{ji} = \frac{G m_i m_j}{|\vec{x}_j - \vec{x}_i|^2} \vec{e}_{ij} = \frac{G m_i m_j}{|\vec{x}_{ij}|^3} \vec{x}_{ij}$$

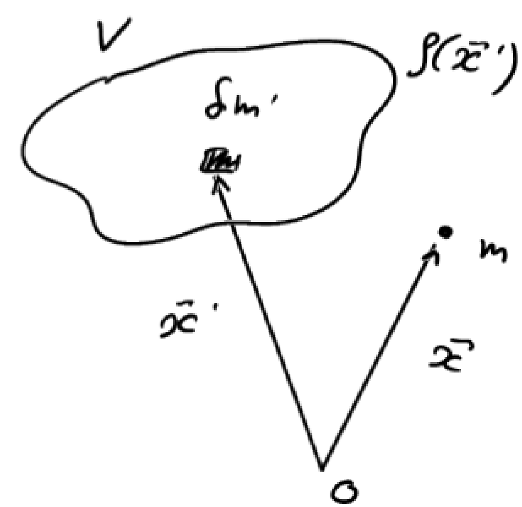



$$\vec{x}_{ij} = \vec{x}_j - \vec{x}_j$$

## Force on a particle of mass $m$ in $\vec{x}$ due to a distribution of mass $\rho(\vec{x})$

$$\delta \vec{F}(\vec{x}) = \frac{G m \, \delta m'}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x})$$

$$= \frac{G m \, \rho(\vec{x}') \, d^3\vec{x}'}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x})$$



So, the total force writes :

$$\vec{F}(\vec{x}) = \int_V \frac{G\, m\, \rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x})\, d^3\vec{x}'$$

$$= m \underbrace{G \int_V \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x})\, d^3\vec{x}'}$$

$\vec{g}(\vec{x})$ : gravitational field

$$[\vec{g}] = \frac{cm}{s^2} = \frac{erg}{g} \frac{1}{cm}$$

# Gravitational Potential

It is easy to see that the function

$$\delta V(\vec{x}) = - \frac{G m \delta m}{|\vec{x}' - \vec{x}|} \qquad \text{is such that}$$

$$\vec{\nabla} \delta V(\vec{x}) = - \frac{G m \delta m}{|\vec{x}' - \vec{x}|^2} \frac{(\vec{x}' - \vec{x})}{|\vec{x}' - \vec{x}|} = - \delta \vec{F}(\vec{x})$$

so, by defining

$$\boxed{V(\vec{x}) = - G \int_V \frac{m \rho(\vec{x}')}{|\vec{x}' - \vec{x}|} d^3\vec{x}'}$$

we ensure that $\quad \vec{\nabla} V(\vec{x}) = - \vec{F}(\vec{x})$

We define the specific potential $\phi(\bar{x}) = \frac{V(\bar{x})}{m}$

which writes

$$\boxed{\phi(\bar{x}) = -G \int_V \frac{\rho(\bar{x}')}{|\bar{x}' - \bar{x}|} \, d^3\bar{x}'}$$

$[\phi] = \frac{erg}{g}$

$\equiv$ specific energy

The gravitational field writes :

$$\boxed{\bar{g}(\bar{x}) = -\bar{\nabla}\phi(\bar{x})}$$

**Notes**

- The gravity is a conservative force

- $\phi(\vec{x})$ : scalar field
  $\vec{g}(\vec{x})$ : vector field
  } contain the same information

- we will always use "specific" quantities

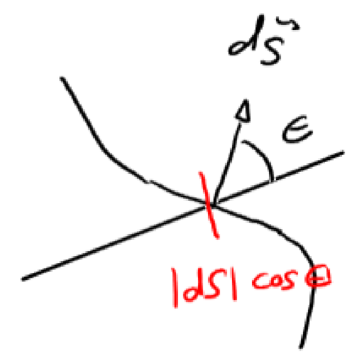
$$V(\vec{x}) \rightarrow \phi(\vec{x})$$

$$K = \frac{1}{2} m \vec{v}^2 \rightarrow \frac{1}{2} \vec{v}^2$$

$$\frac{1}{2} v^2 + \phi(\vec{x}) = \text{specific energy (conserved quantity)}$$

# The Gauss's Law

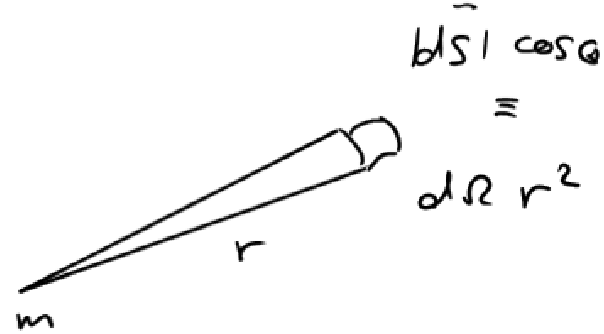

Consider:
- a single point mass m
- a surface S around this point
- a point $\vec{x}$ on the surface at a distance r
- $\vec{g}(\vec{x})$ the gravitational field
- $d\vec{S}$, the normal at the surface
- $\theta$ the angle between $\vec{g}(\vec{x})$ and $d\vec{S}$




$$\vec{g}(\vec{x}) \cdot d\vec{S} = -|\vec{g}(\vec{x})| \cdot |d\vec{S}| \cos\theta$$

But
$$|d\vec{S}| \cos\theta = r^2 \, d\Omega$$

$$|\vec{g}(x)| = \frac{Gm}{r^2}$$



$$\boxed{\vec{g}(\vec{x}) \cdot d\vec{S} = -G m \, d\Omega}$$

integrating over any surface

$$\int_S \vec{g}(\vec{x}) \cdot d\vec{S} = \begin{cases} -4\pi G m & \text{if } m \text{ inside } S \\ 0 & \text{instead} \end{cases}$$

For multiple masses $m_i$

$$\int_S \vec{g}(\vec{x}) \cdot d\vec{S} = -4\pi G \sum_{i \text{ in } S} m_i$$

For a continuous mass distribution $\rho(\vec{x})$

$$\boxed{\int_S \vec{g}(\vec{x}) \cdot d\vec{S} = -4\pi G \int_V \rho(\vec{x}) \, d\vec{x} = -4\pi G M}$$

Gauss's Law

## Divergeance of the specific force (A) $\vec{\nabla}_x \cdot \vec{g}(\vec{x})$

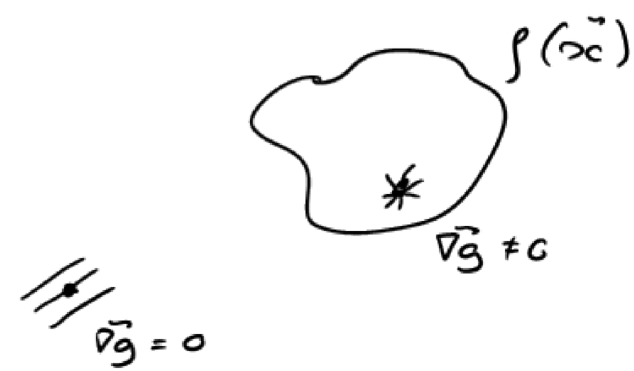
div. theorem

$$\int_V \vec{\nabla} \cdot \vec{g}(\vec{x})\, d^3\vec{x} = \int_S \vec{g}(\vec{x})\, d\vec{S}$$

Gauss's Law

$$= -4\pi G \int_V \rho(\vec{x})\, d\vec{x}$$

$$\boxed{\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = -4\pi G\, \rho(\vec{x})}$$



# The Poisson Equation

$$\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = -4\pi G \rho(\vec{x})$$

with : $\vec{\nabla}_x \phi(\vec{x}) = -\vec{g}(\vec{x})$ $\qquad \vec{\nabla}_x \cdot (\vec{\nabla}_x) = \vec{\nabla}_x^2$

$$\boxed{\vec{\nabla}_x^2 \phi(\vec{x}) = 4\pi G \rho(\vec{x})}$$

Poisson Equation

<u>Note</u> : To ensure a unique solution, boundary conditions are necessary (2nd order diff. eqn.)

<u>ex</u> : $\phi(\infty) = 0$

$\vec{\nabla}\phi(\infty) = \vec{g}(x) = 0$

COMPLEMENT

## Divergence of the specific force (B) $\vec{\nabla}_x \cdot \vec{g}(\vec{x})$

$$\vec{g}(\vec{x}) = G \int_V \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x})\, d^3\vec{x}'$$

$$\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = G \int_V \vec{\nabla}_x \cdot \left( \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x}) \right) d^3\vec{x}'$$

- $$\vec{\nabla}_x \cdot \left( \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) = \frac{d}{dx_1}\left( \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) + \frac{d}{dx_2}\left( \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) + \frac{d}{dx_3}\left( \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right)$$

$$= -\frac{3}{|\vec{x}' - \vec{x}|^3} + \frac{3(\vec{x}' - \vec{x}) \cdot (\vec{x}' - \vec{x})}{|\vec{x}' - \vec{x}|^5}$$

$$= 0 \quad \text{if} \quad \vec{x}' \neq \vec{x}$$

$$\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = G \int_V \vec{\nabla}_x \cdot \left( \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x}) \right) d^3\vec{x}'$$


$\rho(\vec{x}) = \text{cte}$

$$= G \rho(\vec{x}) \int_{|\vec{x}' - \vec{x}| \leq h} \vec{\nabla}_x \cdot \left( \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) d^3\vec{x}'$$

variable exchange

$\vec{\nabla}_x\, f(\vec{x} - \vec{x}') = -\vec{\nabla}_{x'} \cdot f(x - x')$

$$= - G \rho(\vec{x}) \int_{|\vec{x}' - \vec{x}| \leq h} \vec{\nabla}_{x'} \cdot \left( \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) d^3\vec{x}'$$

divergence theorem

$$= - G \rho(\vec{x}) \underbrace{\int_{|\vec{x}' - \vec{x}| = h} \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} d^2\vec{S}'}_{4\pi h^2 \cdot \left. \frac{1}{r^2} \right|_{h=r} = 4\pi}$$

$r = |\vec{x}' - \vec{x}| = h$

$$\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} = \frac{1}{r^2}$$

$$\underline{\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = - 4\pi G \rho(\vec{x})}$$

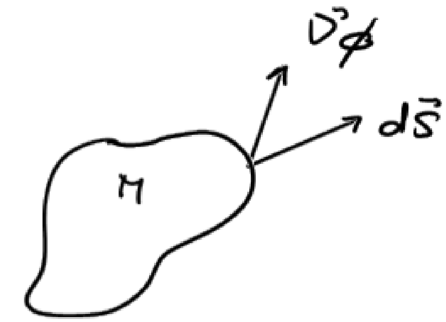
COMPLEMENT

COMPLEMENT

## Gauss theorem

(B) integrate the Poisson equation over a volume $V$ that contains a mass $M$

$$\int_V \vec{\nabla}^2 \phi(\vec{x})\, d^3\vec{x} = \underbrace{\int_V 4\pi G \rho(\vec{x})\, d^3\vec{x}}$$




div. theorem ↓

$$\boxed{\int_S d^2\vec{S} \cdot \vec{\nabla}\phi = 4\pi G M}$$

Gauss theorem

Equivalently :

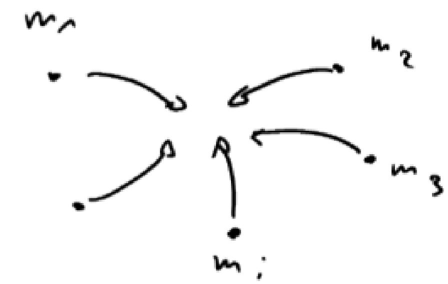
$$\boxed{\int_S d^2\vec{S} \cdot \vec{g}(\vec{x}) = -4\pi G M}$$

Gauss's Law

## Total potential energy (1.0)

Total work needed to assemble a density distribution $\rho(\vec{x})$

$\rho(\vec{x})$ , $\phi(\vec{x})$




Assume a set of discrete points

- The work to bring the 1st point from $\infty$ to $\vec{x}_1$ is $0$
- The work to bring the 2nd point from $\infty$ to $\vec{x}_2$ is $-\dfrac{G m_1 m_2}{r_{12}}$
- The work to bring the 3rd point from $\infty$ to $\vec{x}_3$ is $-\dfrac{G m_1 m_3}{r_{13}} - \dfrac{G m_2 m_3}{r_{23}}$

The total work is thus

$$W = -\frac{G m_1 m_2}{r_{12}} - \frac{G m_1 m_3}{r_{13}} - \frac{G m_2 m_3}{r_{23}} - \ldots - \sum_{j=1}^{N-1} \frac{G m_{jN}}{r_{jN}}$$

$$= -\sum_{i=1}^{N} \sum_{j=1}^{i-1} \frac{G m_i m_j}{r_{ij}} = -\frac{1}{2} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} \frac{G m_i m_j}{r_{ij}}$$

With $\phi_i = -\sum_{\substack{j=1 \\ j \neq i}} \frac{G m_j}{r_{ij}}$ (potential on $i$)

$$W = \frac{1}{2} \sum_{i=1}^{N} m_i \phi_i \equiv \frac{1}{2} \sum_{i=1}^{N} V_i$$

For a continuous mass distribution $\rho(\vec{x})$

$$\boxed{W = \frac{1}{2} \int \rho(\vec{x}) \phi(\vec{x}) \, d^3\vec{x}}$$

## Total potential energy (1.1)

From $$W = \frac{1}{2} \int \rho(\vec{x}) \phi(\vec{x}) \, d^3\vec{x}$$

- replace $\rho(\vec{x})$ with the Poisson equation $\rho(\vec{x}) = \frac{1}{4\pi G} \nabla^2 \phi$

$$W = \frac{1}{8\pi G} \int \nabla^2\phi \cdot \phi(\vec{x}) \, d^3\vec{x} = \frac{1}{8\pi G} \int \vec{\nabla} \cdot (\vec{\nabla}\phi) \cdot \phi(\vec{x}) \, d^3\vec{x}$$

- divergence theorem $\int d^3x \; g \cdot \vec{\nabla} \cdot \vec{F} = \int_S g \cdot \vec{F} \, d\vec{S} - \int d^3x \; \vec{F} \cdot \vec{\nabla} g$

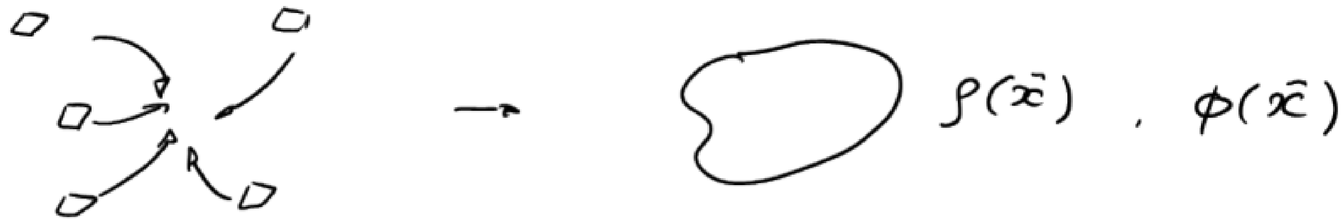
$$W = \frac{1}{8\pi G} \left[ \underbrace{\int \phi \, \vec{\nabla}\phi \, d\vec{S}}_{=0} - \int d^3\vec{x} \; \vec{\nabla}\phi \cdot \vec{\nabla}\phi \right]$$

$= 0$ as $\phi(\infty) = \vec{\nabla}\phi(\infty) = 0$

$$\boxed{W = -\frac{1}{8\pi G} \int d^3\vec{x} \, |\vec{\nabla}\phi|^2}$$

# Total potential energy (2.0)

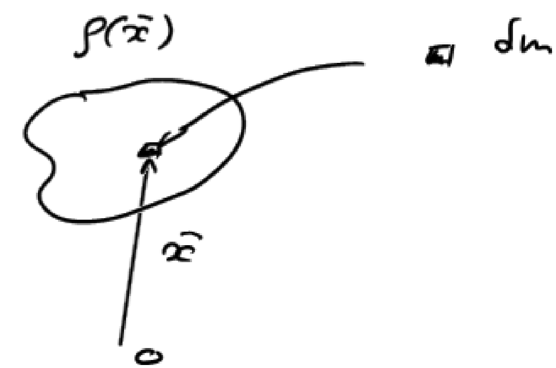
Total work needed to assemble a density distribution $\rho(\vec{x})$




(1) Work done to assemble a piece of mass $\delta m = \delta\rho \, d\vec{x}^3$ from $\infty$ to $\vec{x}$ assuming an existing mass distribution $\rho(\vec{x})$, $\phi(\vec{x})$

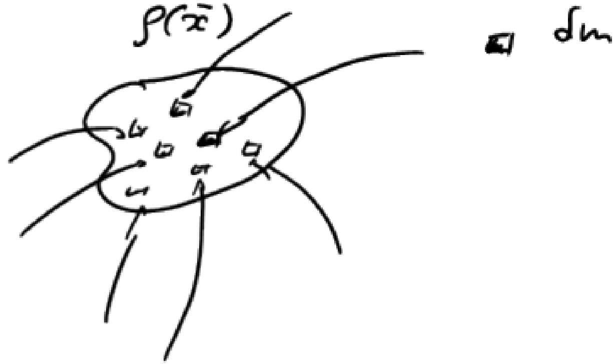
$$\delta W_{\vec{x}} = V(\vec{x}) - \underbrace{V(\infty)}_{=0}$$

$$= \delta m \, \phi(\vec{x}) = \delta\rho(\vec{x}) \, d^3\vec{x} \; \phi(\vec{x})$$



To increase everywhere the mass distribution by $\delta\rho$

$$\rho(\vec{x}) \to \rho(\vec{x}) + \delta\rho(\vec{x})$$




$$\delta W = \int \delta\rho(\vec{x})\, d^3\vec{x}\ \phi(\vec{x})$$

Poisson: $\delta\rho(\vec{x}) = \frac{1}{4\pi G} \vec{\nabla}^2 \delta\phi(x)$

$$\delta W = \frac{1}{4\pi G} \int \vec{\nabla}^2 \delta\phi(x)\ \phi(\vec{x})\ d^3\vec{x}$$

divergeance theorem

$$\int_V d^3x\, g\, \vec{\nabla}\cdot\vec{F} = \int_S g\, \vec{F}\cdot d^2\vec{S} - \int_V d^3x\, \vec{F}\cdot\vec{\nabla} g$$

$$= \frac{1}{4\pi G} \underbrace{\int_{S \text{ at } \infty} \phi(\vec{x})\, \vec{\nabla}\delta\phi(\vec{x})}_{=0} - \frac{1}{4\pi G} \int \vec{\nabla}\phi(\vec{x}) \cdot \vec{\nabla}(\delta\phi(\vec{x}))\ d^3\vec{x}$$

as $\phi(\infty) = 0$

$\vec{\nabla}\delta\phi(\infty) = \delta g(\infty) = 0$

COMPLEMENT

$$\delta W = -\frac{1}{4\pi G}\int \vec{\nabla}\phi(\vec{x})\cdot\vec{\nabla}(\delta\phi(\vec{x}))\, d^3\vec{x}$$

with

$$\frac{1}{2}\,\delta|\vec{\nabla}\phi(\vec{x})|^2 = \delta\vec{\nabla}\phi(\vec{x})\cdot\vec{\nabla}\phi(\vec{x}) = \vec{\nabla}(\delta\phi(\vec{x}))\cdot\vec{\nabla}\phi(\vec{x})$$

$$\delta W = -\frac{1}{8\pi G}\int \delta|\vec{\nabla}\phi|^2 d^3x \equiv -\frac{1}{8\pi G}\,\delta\int |\vec{\nabla}\phi|^2 d^3x$$

② Contribution of all $\delta W$ to $W$

$$\boxed{W = -\frac{1}{8\pi G}\int |\vec{\nabla}\phi|^2 d^3x}$$

## Total potential energy (2.2)

From $W = -\frac{1}{8\pi G}\int |\vec{\nabla}\phi|^2 d^3x = -\frac{1}{8\pi G}\int \vec{\nabla}\phi \cdot \vec{\nabla}\phi \, d^3\vec{x}$

- divergence theorem $\int d^3x \, \vec{F}\cdot\vec{\nabla} g = \int_S g \, \vec{F} \cdot d\vec{S} - \int d^3x \, g \, \vec{\nabla}\vec{F}$

$$W = -\frac{1}{8\pi G}\left[\underbrace{\int_S \phi \vec{\nabla}\phi \, d\vec{S}}_{=0 \text{ as } \phi(\infty) = \vec{\nabla}\phi(\infty) = 0} - \int d^3\vec{x} \; \phi \; \vec{\nabla}\underbrace{(\vec{\nabla}\phi)}_{4\pi G \rho \text{ (Poisson)}}\right]$$

$$= \frac{1}{8\pi G} 4\pi G \int d^3\vec{x} \; \phi(\vec{x}) \, \rho(\vec{x})$$

$$\boxed{W = \frac{1}{2}\int \rho(\vec{x})\,\phi(\vec{x})\, d^3\vec{x}}$$

COMPLEMENT

# Total potential energy : Summary

$$W = \frac{1}{2} \int \rho(\vec{x})\, \phi(\vec{x})\, d^3\vec{x}$$

$$W = -\frac{1}{8\pi G} \int |\vec{\nabla} \phi|^2\, d^3x$$

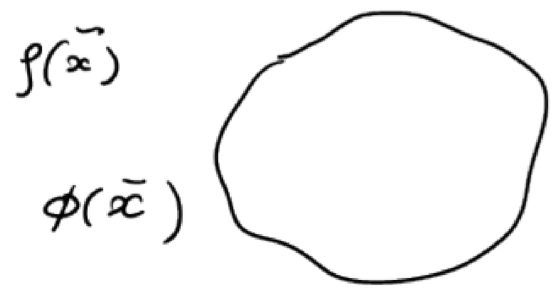
Other useful expression

$$W = -\int \rho(\vec{x})\, \vec{x} \cdot \vec{\nabla} \phi(\vec{x})\, d^3\vec{x}$$

# Relation between the potential energy and the Poisson equation

---

What is the relation that must hold between the density $\rho(\bar{x})$ and potential $\phi(\bar{x})$ in order to minimize the potential energy of a system?




W : potential energy

<u>Answer</u> : the Poisson equation $\nabla^2 \phi = 4\pi G \rho$

# Spherical Systems

$$\rho(\vec{x}) = \rho(r)$$

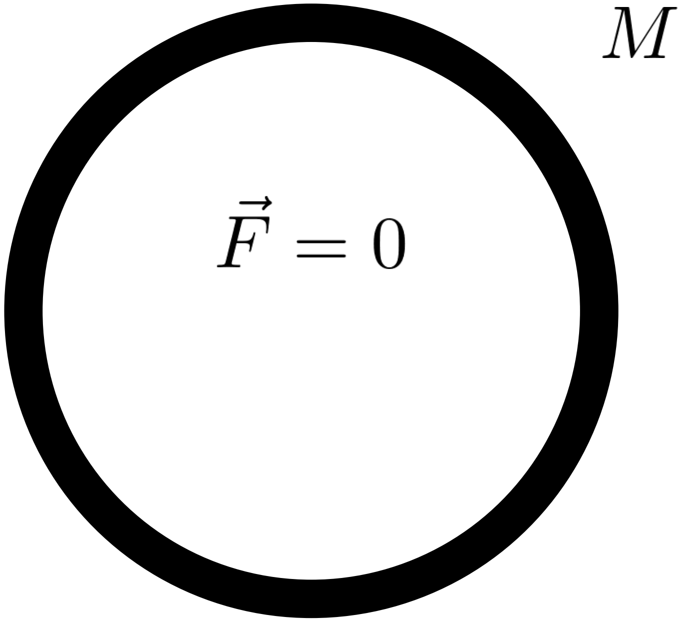
$$r = \sqrt{x^2 + y^2 + z^2}$$

# Newton's Theorems

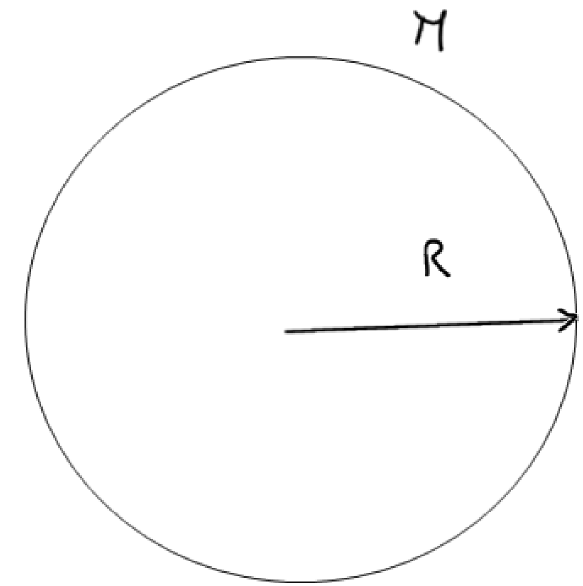
Newton (1642-1727)

First theorem:

A body that is inside a spherical shell of matter experiences no net gravitational force from the shell.



# Spherical infinitly thin shell




Radius : $R$

Mass : $M$

Density : $\rho(r) = \frac{M}{4\pi r^2} \delta(R-r)$

indeed :

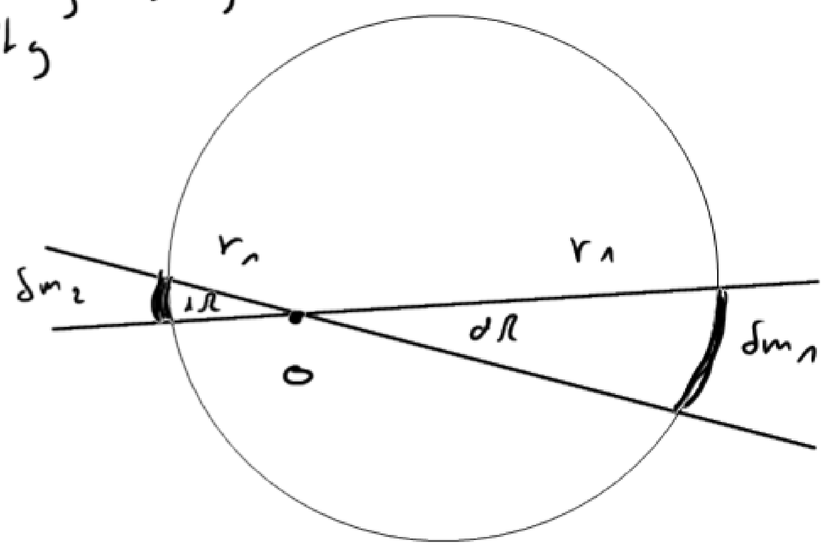
$$M := 4\pi \int_0^\infty dr\, r^2 \rho(r)$$

$$= 4\pi \int_0^\infty dr\, r^2 \frac{M}{4\pi r^2} \delta(R-r) = M$$

# First Newton theorem

A body that is inside a spherical shell of matter experiences no net gravitational force from that shell

a shell of constant density $\rho(\vec{x}) = \rho$




$$\begin{cases} \delta m_1 = \rho(r_1) \cdot r_1^2 \, d\Omega \, dr \\ \delta m_2 = \rho(r_2) \cdot r_2^2 \, d\Omega \, dr \end{cases}$$

thus : $$\frac{\delta m_1}{\delta m_2} = \frac{r_1^2}{r_2^2}$$

and $$\frac{\delta m_1}{r_1^2} = \frac{\delta m_2}{r_2^2}$$

consequently : $\delta \vec{F_1} = -\delta \vec{F_2}$

by integrating over the entire shell ($d\Omega$)

all forces cancel out ! #

**Corollary** The gravitational potential $\phi(\bar{x})$ is constant inside the sphere.

As $\vec{\nabla}_x \phi(\bar{x}) = \bar{g} = 0$ $\qquad \phi(\bar{x}) = \text{ct}$ #

**What is the value of $\phi(\bar{x})$ ?**

$$\phi(\bar{x}) = -\int_V \frac{G\,\rho(\bar{x}')}{|\bar{x}' - \bar{x}|}\, d^3\bar{x}'$$

**Spherical coordinates** $\qquad d^3\bar{x} = r^2\,dr\,d\Omega = 4\pi\, r^2 dr$

**At the center** $\quad \bar{x} = 0$

$$\phi(0) = -4\pi G \int_0^\infty \frac{\rho(r')}{r'}\, r'^2 dr' = -4\pi G \int_0^\infty \rho(r')\, r' dr'$$

with :
$$\rho(r') = \frac{M}{4\pi r'^2} \delta(R-r')$$

$$\phi(r) = -GM \int_0^{\infty} \frac{\delta(R-r)}{r^2} \, r \, dr = -\frac{GM}{R}$$

As the potential is constant for $r < R$

$$\boxed{\phi(\bar{x}) = -\frac{GM}{R} \qquad \bar{x} \in \text{sphere}}$$

# Newton's Theorems

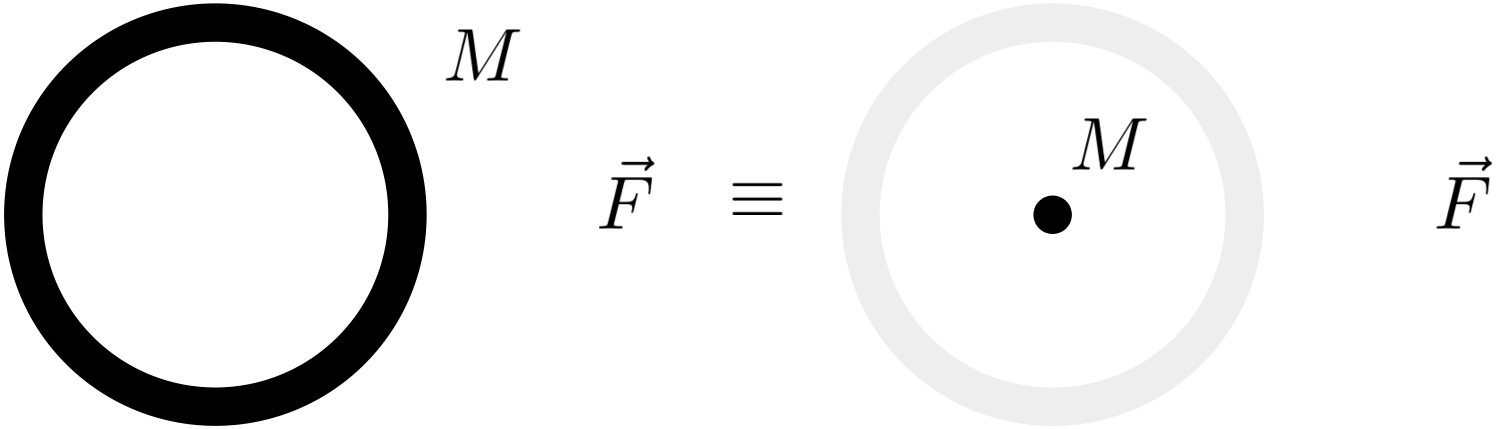
Newton (1642-1727)

First theorem:

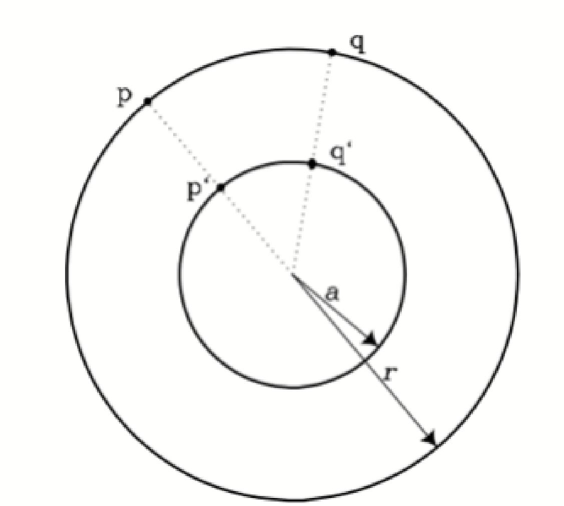
A body that is inside a spherical shell of matter experiences no net gravitational force from the shell.

Second theorem:

The gravitational force on a body that lies outside a spherical shell of matter is the same as it would be if all the shell's matter where concentrated into a point at its centre.



# Second Newton theorem



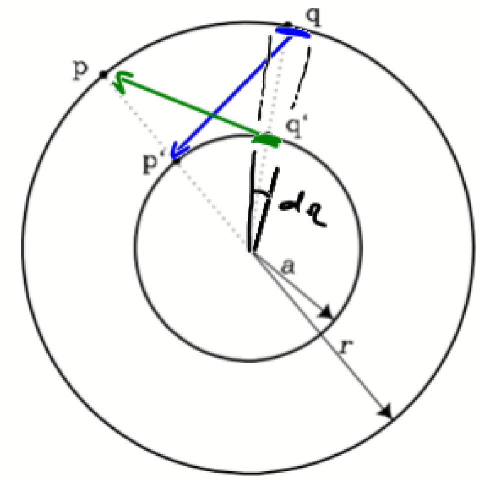

The gravitational force on a body that lies outside a spherical shell of matter is the same as it would be if all the shell's matter were concentrated into a point at its center

## Consider two shells

1. inner, with radius $a$ and mass $M$
2. outer, with radius $r$ and mass $M$

Compute 1. $\phi_p = \phi_i(r)$

2. $\phi_{p'} = \phi_o(a) = -\dfrac{GM}{r}$




1. contribution of shell$_i$, in q' with solid angle $\delta\Omega$

- $$\delta\phi_i(p) = -\frac{G\,\delta m_{q'}}{|p-q'|} = -\frac{GM}{|p-q'|}\frac{\delta\Omega}{4\pi}$$

mass inside the solid angle

$$\delta m = \frac{M\,\delta\Omega}{4\pi}$$

2. contribution of shell$_o$, in q with solid angle $\delta\Omega$

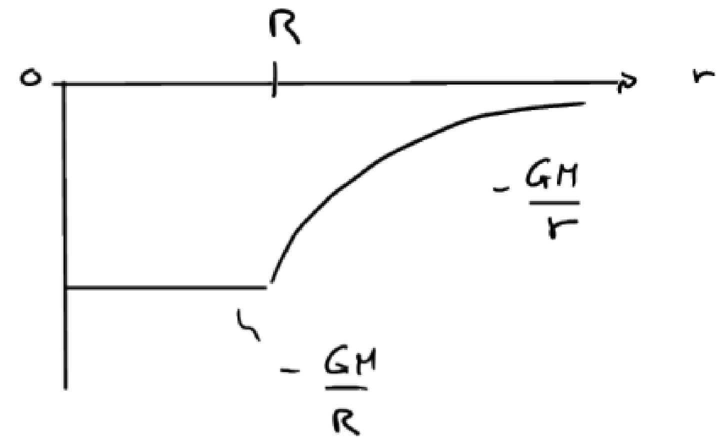
- $$\delta\phi_o(p') = -\frac{G\,\delta m_q}{|p'-q|} = -\frac{GM}{|p'-q|}\frac{\delta\Omega}{4\pi} = \delta\phi_i(p)$$

Summing over all q' $\equiv$ Summing over all q

$$\phi_i(p) = \phi_o(p') = -\frac{GM}{r}$$ #

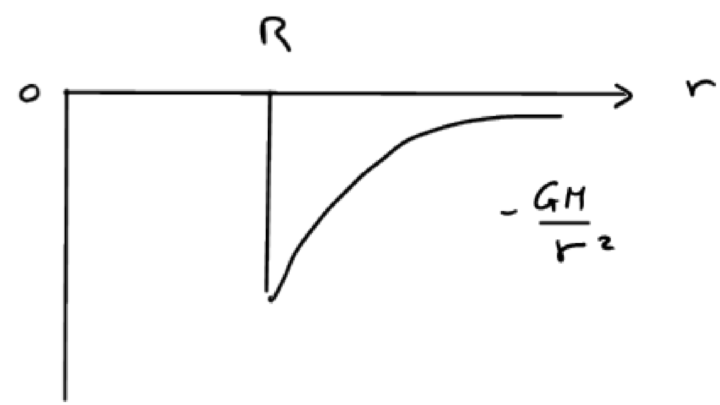
## Total potential of a shell of mass M, radius R

$$\phi(r) = \begin{cases} -\dfrac{GM}{R} & r < R \\ -\dfrac{GM}{r} & r \geq R \end{cases}$$




## Total gravitational field of a shell of mass M, radius R

$$\vec{g}(r) = \begin{cases} 0 & r < R \\ -\dfrac{GM}{r^2}\,\vec{e}_r & r \geq R \end{cases}$$



# Spherical Systems
# general distribution of mass

$$\rho(\vec{x}) = \rho(r)$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

# Potential and gravitational field of any density $\rho(r)$

Build any density by summing shells of of size $R$, mass $M_R$ and density $\rho_R(r)$

$$\rho(r) = \sum_R \rho_R(r) = \int dR \frac{\partial \rho_R(r)}{\partial R}$$

$$= \int_0^\infty dR \frac{\partial M_R}{\partial R} \frac{1}{4\pi r^2} \delta(R-r) = \frac{\partial M_r}{\partial r} \frac{1}{4\pi r^2}$$

mass per unit length

$$\boxed{\partial M_R = 4\pi R^2 \rho(R)\, dR}$$

Each shell contributing to the total density has thus a potential

$$\delta\phi_R(r) = \begin{cases} \dfrac{G\ 4\pi R^2\ \rho(R)\ dR}{R} & r < R \\ \dfrac{G\ 4\pi R^2\ \rho(R)\ dR}{r} & r \geq R \end{cases}$$

$$\delta\phi_R(r) = \begin{cases} -4\pi G R\ \rho(R)\ dR & r < R \\ -\dfrac{4\pi G\ R^2\ \rho(R)\ dR}{r} & r \geq R \end{cases}$$

## Total Potential

$$\phi(r) = \int_0^\infty \delta\phi_R(r)$$

$$= \underbrace{\int_0^r \delta\phi_R(r)}_{\text{inner shells } r \geq R} + \underbrace{\int_r^\infty \delta\phi_R(r)}_{\text{outer shells } r < R}$$

$$= \underbrace{-4\pi G \int_0^r \frac{R^2 \rho(R)}{r} dR}_{\frac{GM(r)}{r}} - 4\pi G \int_r^\infty R\,\rho(R)\,dR$$

$$\phi(r) = -\frac{GM(r)}{r} - 4\pi G \int_r^{\infty} dR\, R\, \rho(R)$$

$$\underbrace{-\frac{GM(r)}{r}}_{\text{contribution of the mass inside } r} \qquad \underbrace{- 4\pi G \int_r^{\infty} dR\, R\, \rho(R)}_{\text{contribution of the mass outside } r}$$

# Gravitational field of a spherical model $\rho(r)$

From the potential $\phi(r)$ $\quad \vec{g}(\vec{x}) = -\vec{\nabla}\phi(\vec{x})$

$$g(r) = -\frac{\partial \phi}{\partial r} = -\frac{\partial}{\partial r}\left[-\frac{GM(r)}{r} - 4\pi G \int_r^{\infty} \rho(r')\, r'\, dr'\right]$$

$$= -\frac{GM(r)}{r^2} + \frac{G}{r} 4\pi \frac{\partial}{\partial r}\int_0^r dr'\, r'^2 \rho(r') + 4\pi G \frac{\partial}{\partial r}\int_r^{\infty} dr'\, r'\, \rho(r')$$

$$= -\frac{GM(r)}{r^2} + \underbrace{\frac{G}{r} 4\pi\, r^2 \rho(r) - 4\pi G\, r\, \rho(r)}_{=0}$$

$$\boxed{g(r) = -\frac{GM(r)}{r^2}}$$

contribution of the mass inside $r$

# Gravitational field of a spherical model $\rho(r)$

## Sum of shells

$$g(r) = \int_0^\infty \delta g_{r'}(r)$$

$\delta g_{r'}(r)$ = force due to the shell of radius $r'$

$$= \underbrace{\int_0^r \delta g_{r'}(r)}_{\text{inner shells}} + \underbrace{\int_r^\infty \delta g_{r'}(r)}_{\text{outer shells}}$$

outer shells = 0 as we are inside

### mass of a shell

$$\delta M(r') = 4\pi r'^2 dr' \rho(r') \qquad \delta g_{r'}(r) = -\frac{G\,\delta M(r')}{r^2} = -4\pi \rho(r') \frac{r'^2}{r^2} dr'$$

$$g(r) = -\frac{G}{r^2} \underbrace{4\pi \int_0^r \rho(r') r'^2 dr'}_{M(r)} = -\frac{G M(r)}{r^2}$$

## Summary: for any spherical mass distribution $\rho(r)$

$$g(r) = -\frac{G M(r)}{r^2}$$

$$M(r) = 4\pi \int_0^{\infty} \rho(r')\, r'^2\, dr'$$

$$\phi(r) = -\frac{G M(r)}{r} - 4\pi G \int_r^{\infty} \rho(r')\, r'\, dr'$$

<u>Note</u>

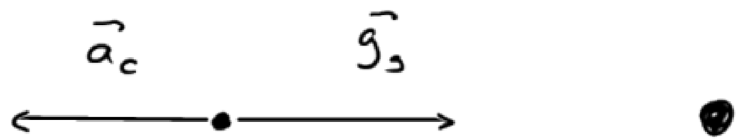
$$g(r) = -\frac{\partial \phi}{\partial r}$$

as expected from

$$\bar{g}(\bar{x}) = \bar{\nabla} \phi(\bar{x})$$

# Spherical systems : circular speed, circular velocity

Speed of a test particle in a cirular orbit in the potential $\phi(r)$ at a radius $r$ :




$\vec{a}_c$ : centrifugal acceleration $\qquad \frac{V_c^2}{r}$

$\vec{g}_s$ : gravity acceleration ( spec force ) $\qquad -\frac{GM(r)}{r^2} = -\frac{\partial\phi}{\partial r}$

$$V_c^2 = \frac{GM(r)}{r}$$

$$V_c^2 = r\,\frac{\partial\phi}{\partial r}$$

$$GM(r) = r^2\,\frac{\partial\phi}{\partial r}$$

$[V_c^2] : \frac{erg}{g}$

as $\phi$

= specific energy

## Velocity composition

**Note**: $V_0^2$ scale with the mass ($M(r)$) : it is thus the "important" quantity (spec. energy)

Multi-components system : ex: bulge + stellar halo + DM halo

$$\begin{cases} \rho_B(r) \,,\; M_B(r) \,,\; \phi_B(r) \longrightarrow V_{c,B}(r) \\ \rho_H(r) \,,\; M_H(r) \,,\; \phi_H(r) \longrightarrow V_{c,H}(r) \\ \rho_{DM}(r) \,,\; M_{DM}(r) \,,\; \phi_{DM}(r) \longrightarrow V_{c,DM}(r) \end{cases}$$

$$V_{c,tot}^2 = \frac{G M_{tot}(r)}{r} = \frac{G}{r} \sum_i M(r)$$

$$\boxed{V_{c,tot}^2 = \sum_i V_{c,i}^2}$$

$V_c^2 \sim$ energy : extensive quantity

## Period of the circular orbit

$$T(r) = \frac{2\pi r}{V_c(r)} = 2\pi\sqrt{\frac{r^3}{GM(r)}} = 2\pi\sqrt{\frac{r}{\frac{\partial\phi}{\partial r}}}$$

## Circular frequency (angular frequency)

$$\Omega(r) = \frac{2\pi}{T(r)} = \sqrt{\frac{GM(r)}{r^3}} = \sqrt{\frac{1}{r}\frac{\partial\phi}{\partial r}}$$

## Escape speed $V_e$

if $\frac{1}{2}V_e^2 > \phi(r) = E > 0$

the partcle may escape the system

$$V_e(r) = \sqrt{2|\phi(r)|}$$

## Potential energy

from $W = - \int \rho(\vec{x})\, \vec{x} \cdot \vec{\nabla} \phi(\vec{x})\, d^3\vec{x}$

$$W = -4\pi G \int_0^\infty \rho(r)\, M(r)\, r\, dr$$

## Gravitational radius

radius at which $\frac{GM^2}{r} = W$

(estimation of the system size)

$$r_g = \frac{GM^2}{|W|}$$

# Spherical systems : useful relations

| | $\rho(r)$ | $\Phi(r)$ | $M(r)$ | $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$ |
|---|---|---|---|---|
| $\rho(r)$ | $\rho(r)$ | $\frac{1}{4\,\pi\,G}\,\frac{1}{r^2}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)$ | $\frac{1}{4\pi r^2}\,\frac{\mathrm{d}M(r)}{\mathrm{d}r}$ | $\frac{1}{4\,\pi\,G}\,\frac{1}{r^2}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)$ |
| $\Phi(r)$ | $-\frac{G\,M(r)}{r} - 4\pi G\,\int_r^\infty \mathrm{d}r'\,r'\,\rho(r')$ | $\Phi(r)$ | $-G\,\int_r^\infty \mathrm{d}r'\,\frac{M(r')}{r'^2}$ | $-\int_r^\infty \mathrm{d}r'\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}$ |
| $M(r)$ | $4\pi\,\int_0^r \mathrm{d}r'\,r'^2\,\rho(r')$ | $\frac{r^2}{G}\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}$ | $M(r)$ | $\frac{r^2}{G}\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}$ |
| $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$ | $\frac{4\pi G}{r^2}\,\int_0^r \mathrm{d}r'\,r'^2\,\rho(r')$ | $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$ | $\frac{G\,M(r)}{r^2}$ | $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$ |

Poisson in spherical coordinates

$$\frac{1}{r^2}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\,\pi\,G\,\rho(r)$$

Potential in spherical coordinates

$$\Phi(r) = -\frac{G\,M(r)}{r} - 4\pi G\int_r^\infty \rho(r')r'\mathrm{d}r'$$

Mass inside a radius r

$$M(r) = 4\pi\,\int_0^r dr'\,r'^2\,\rho(r')$$

Gradient of the potential in spherical coordinates

$$\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r} = \frac{G\,M(r)}{r^2}$$

# Examples of Spherical models

# "Potential based" models

# Point mass

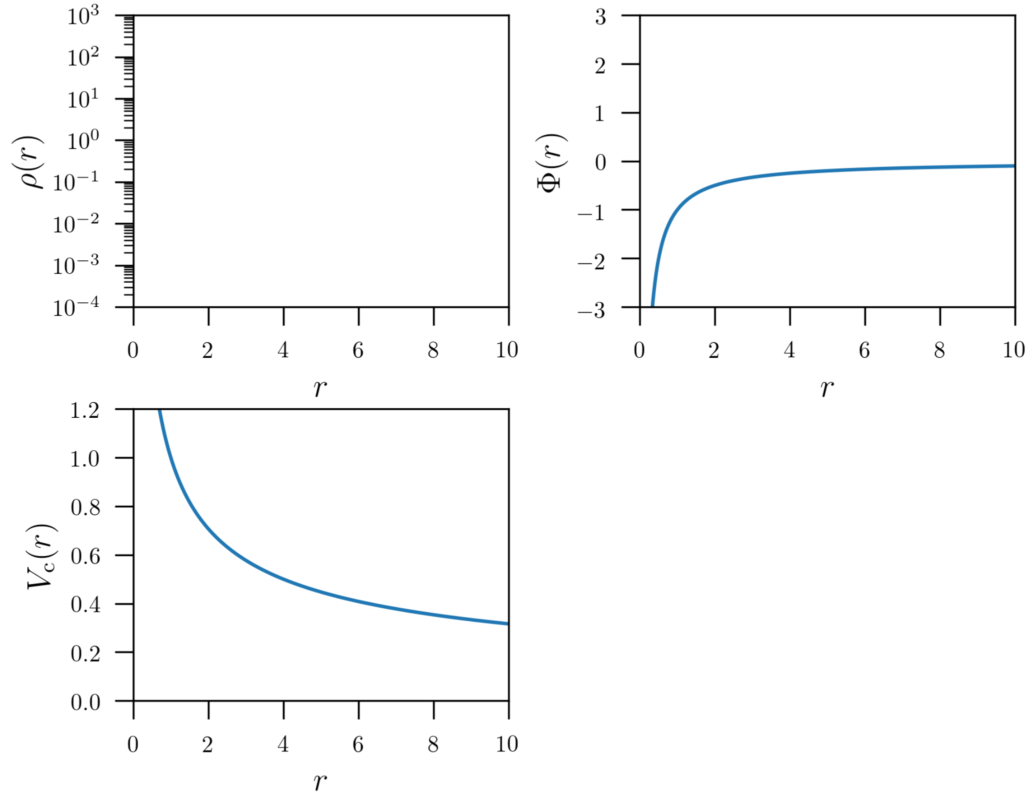
$$\Phi(r) = -\frac{GM}{r}$$

$$\rho(r) = \frac{M\delta(0)}{4\pi r^2}$$

$$M(r) = M$$

$$V_c^2(r) = \frac{GM}{r}$$

$$T(r) = 2\pi\sqrt{\frac{r^3}{GM}}$$



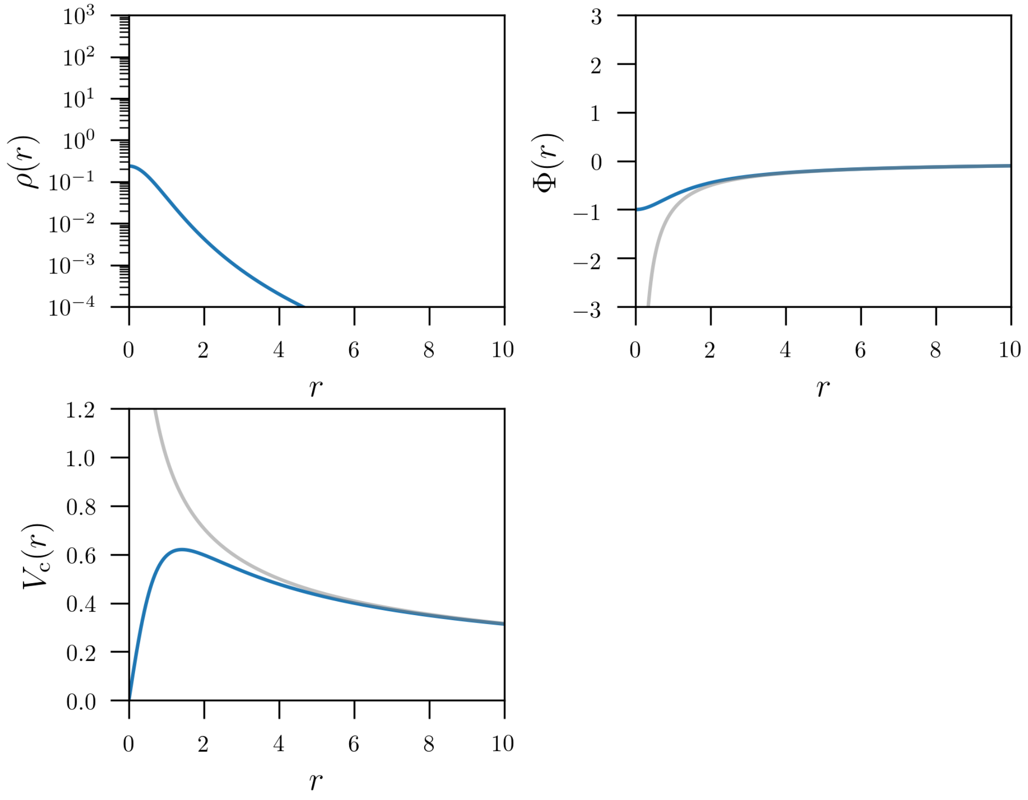
# Plummer model

$$\Phi(r) = -\frac{GM}{\sqrt{r^2 + b^2}}$$

$$\rho(r) = \frac{3M}{4\pi b^3}\left(1 + \frac{r^2}{b^2}\right)^{-5/2}$$

$$M(r) = \frac{Mr^3}{(r^2 + b^2)^{3/2}}$$

$$V_c^2(r) = \frac{GMr^2}{(r^2 + b^2)^{3/2}}$$




- Globular clusters, dwarf spheroidal galaxies

# Isochrone potential

$$\Phi(r) = -\frac{GM}{b + \sqrt{r^2 + b^2}}$$

$$\rho(r) = M\frac{3(b + \sqrt{b^2 + r^2})(b^2 + r^2) - r^2(b + 3\sqrt{b^2 + r^2})}{4\pi(b + \sqrt{b^2 + r^2})^3(b^2 + r^2)^{3/2}}$$

$$M(r) = \frac{Mr^3}{\sqrt{b^2 + r^2}(b + \sqrt{b^2 + r^2})^2}$$

$$V_c^2(r) = \frac{GMr^2}{\sqrt{b^2 + r^2}(b + \sqrt{b^2 + r^2})^2}$$

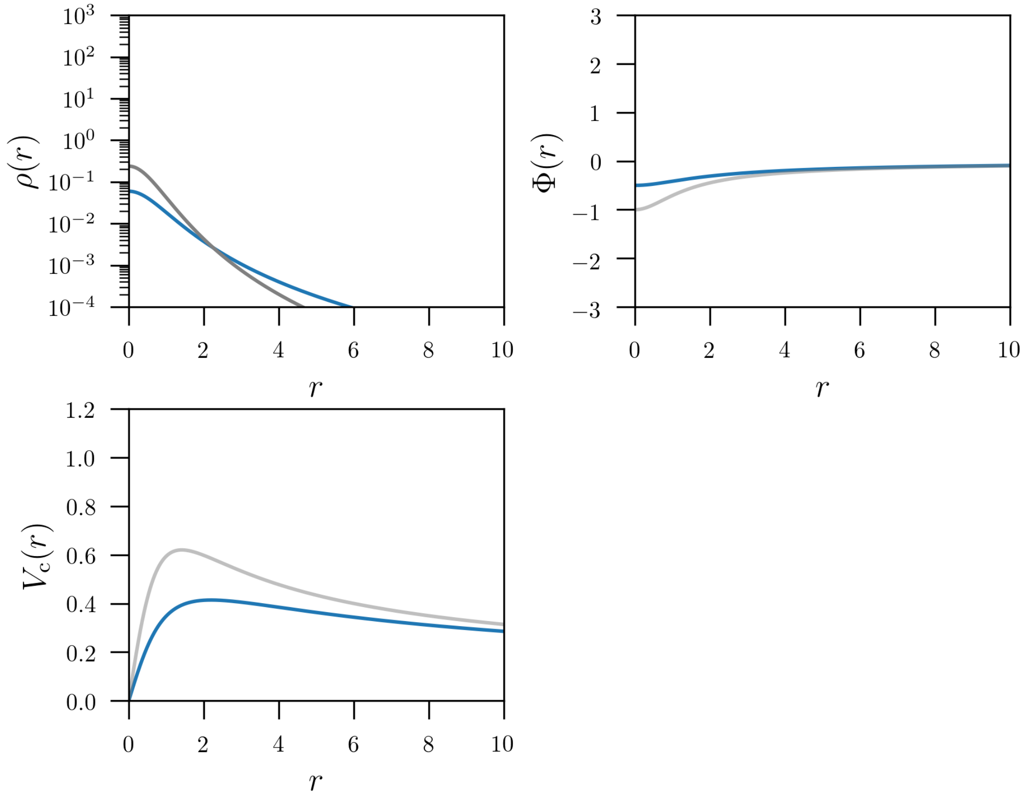
# Isochrone potential

$$\Phi(r) = -\frac{GM}{b + \sqrt{r^2 + b^2}}$$

$$\rho(r) = M\,\frac{3(b + \sqrt{b^2 + r^2})(b^2 + r^2) - r^2(}{4\pi(b + \sqrt{b^2 + r^2})^3(b^2 -}$$

$$M(r) = \frac{Mr^3}{\sqrt{b^2 + r^2}(b + \sqrt{b^2 + r^2})^2}$$

$$V_c^2(r) = \frac{GMr^2}{\sqrt{b^2 + r^2}(b + \sqrt{b^2 + r^2})^2}$$




Orbits are analytical !

# Examples of Spherical models

## “Density based” models

# Homogeneous sphere

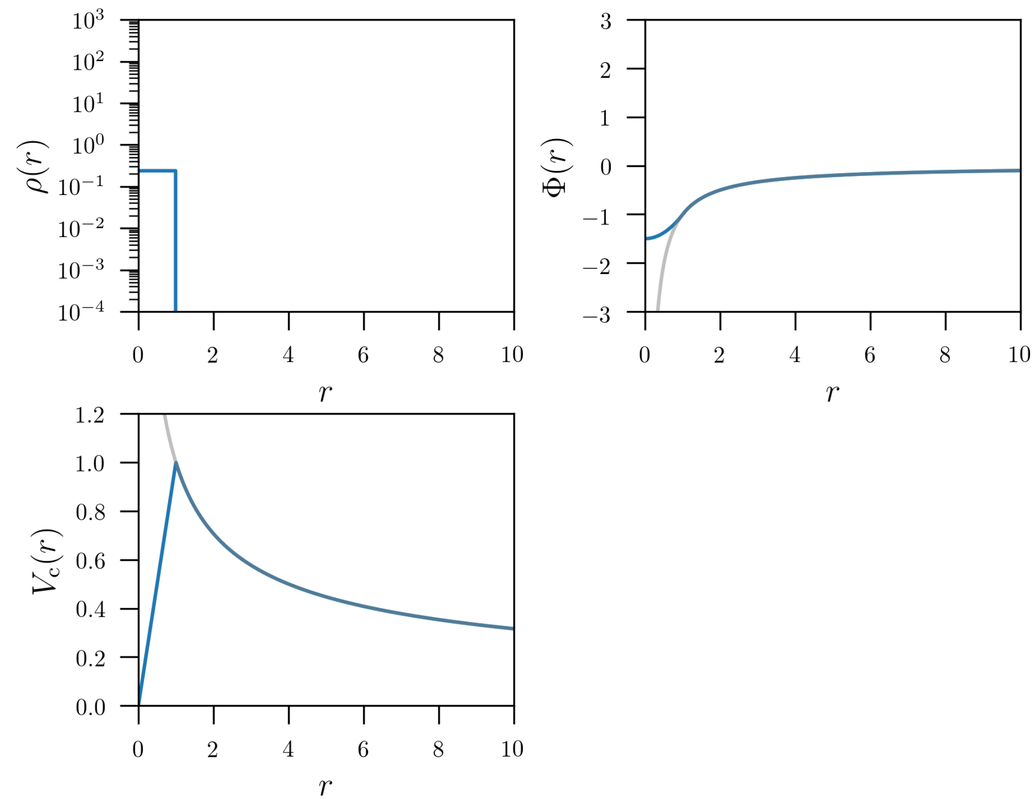
$$\rho(r) = \begin{cases} \rho & r < R \\ 0 & r > R \end{cases}$$

$$M(r) = \begin{cases} \frac{4}{3}\pi r^3 \rho_0 & r < R \\ \frac{4}{3}\pi R^3 \rho & r > R \end{cases}$$

$$\Phi(r) = \begin{cases} -2\pi G\rho \left(R^2 - \frac{1}{3} r^2\right) & r < R \\ -4\pi G\rho R^3/(3r) & r > R \end{cases}$$

$$V_c^2(r) = \begin{cases} \frac{4}{3}\pi G\rho_0 r^2 & r < R \\ \frac{4}{3}\pi G\rho_0 \frac{R^3}{r} & r > R \end{cases}$$

$$T(r) = \begin{cases} \sqrt{\frac{3\pi}{G\rho_0}} & r < R \\ \sqrt{\frac{3\pi}{G\rho_0 R^3}} r^{3/2} & r > R \end{cases}$$




$$\frac{d^2 r}{dt^2} = -\frac{d\Phi(r)}{dr} = -\frac{GM(r)}{r^2} = -\frac{4}{3}\pi\rho_0 r = -\omega^2 r$$

Harmonic oscillator !

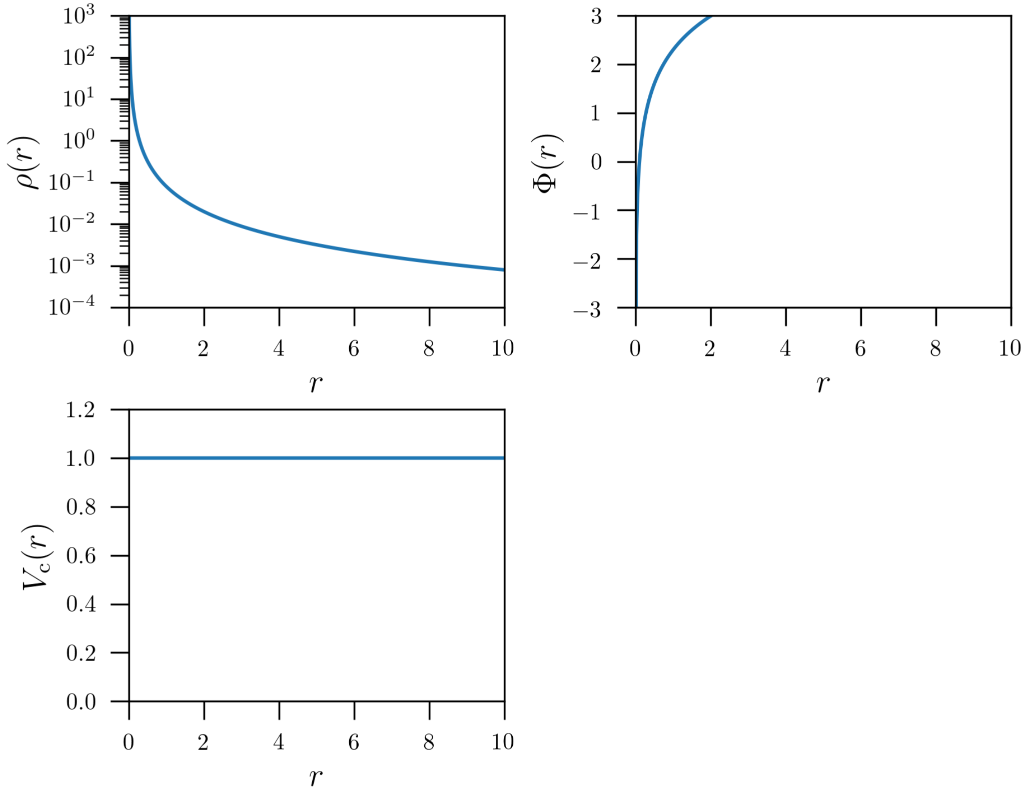
# Isothermal sphere

$$\rho(r) = \rho_0 \frac{a^2}{r^2}$$

$$\Phi(r) = 4\pi G \rho_0 a^2 \ln\left(\frac{r}{a}\right)$$

$$M(r) = 4\pi \rho_0 a^2\, r$$

$$V_c^2(r) = 4\pi G \rho_0 a^2$$




- often used for gravitational lens models
- But !
  - diverge towards the centre !
  - infinite mass !

# Pseudo-isothermal sphere

$$\rho(r) = \rho_0 \frac{a^2}{a^2 + r^2}$$

$$\Phi(r) = 4\pi G \rho_0 a^2 \left( \frac{1}{2} \ln(a^2 + r^2) + \frac{a}{r} \arctan\left(\frac{r}{a}\right) \right)$$

$$M(r) = 4\pi r \rho_0 a^2 \left(1 - \frac{a}{r} \arctan\left(\frac{r}{a}\right)\right)$$

$$V_c^2(r) = 4\pi G \rho_0 a^2 \left(1 - \frac{a}{r} \arctan\left(\frac{r}{a}\right)\right)$$

- Avoid the central divergence of the isothermal sphere
  - However, the mass is still not bounded

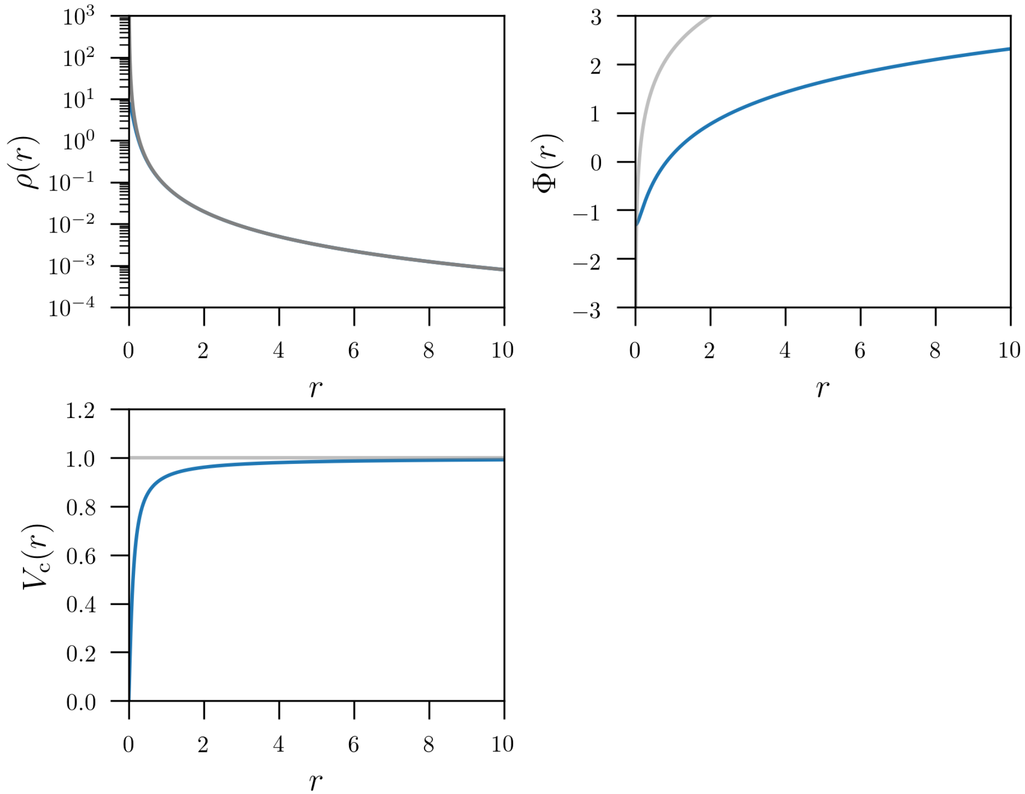
# Pseudo-isothermal sphere

$$\rho(r) = \rho_0 \frac{a^2}{a^2 + r^2}$$

$$\Phi(r) = 4\pi G \rho_0 a^2 \left( \frac{1}{2} \ln(a^2 + r^2) + \frac{a}{r} \,\mathrm{arcta}\right.$$

$$M(r) = 4\pi r \rho_0 a^2 \left( 1 - \frac{a}{r} \arctan\left(\frac{r}{a}\right)\right)$$

$$V_c^2(r) = 4\pi G \rho_0 a^2 \left( 1 - \frac{a}{r} \arctan\left(\frac{r}{a}\right)\right)$$




- Avoid the central divergence of the isothermal sphere
  - However, the mass is still not bounded

# Generic two power density models

$$\rho(r) = \frac{\rho_0}{(r/a)^\alpha (1 + r/a)^{\beta-\alpha}}$$

- diverges at the center if $\alpha \neq 0$

$$M(r) = 4\pi\rho_0 a^3 \int_0^{r/a} s \frac{s^{2-\alpha}}{(1+s)^{\beta-\alpha}}$$

| model name | inner slope $\alpha$ | outer slope $\beta$ |
|---|---|---|
| Plummer | 0 | 5 |
| Dehnen | any | 4 |
| Hernquist | 1 | 4 |
| Jaffe | 2 | 4 |
| NFW | 1 | 3 |

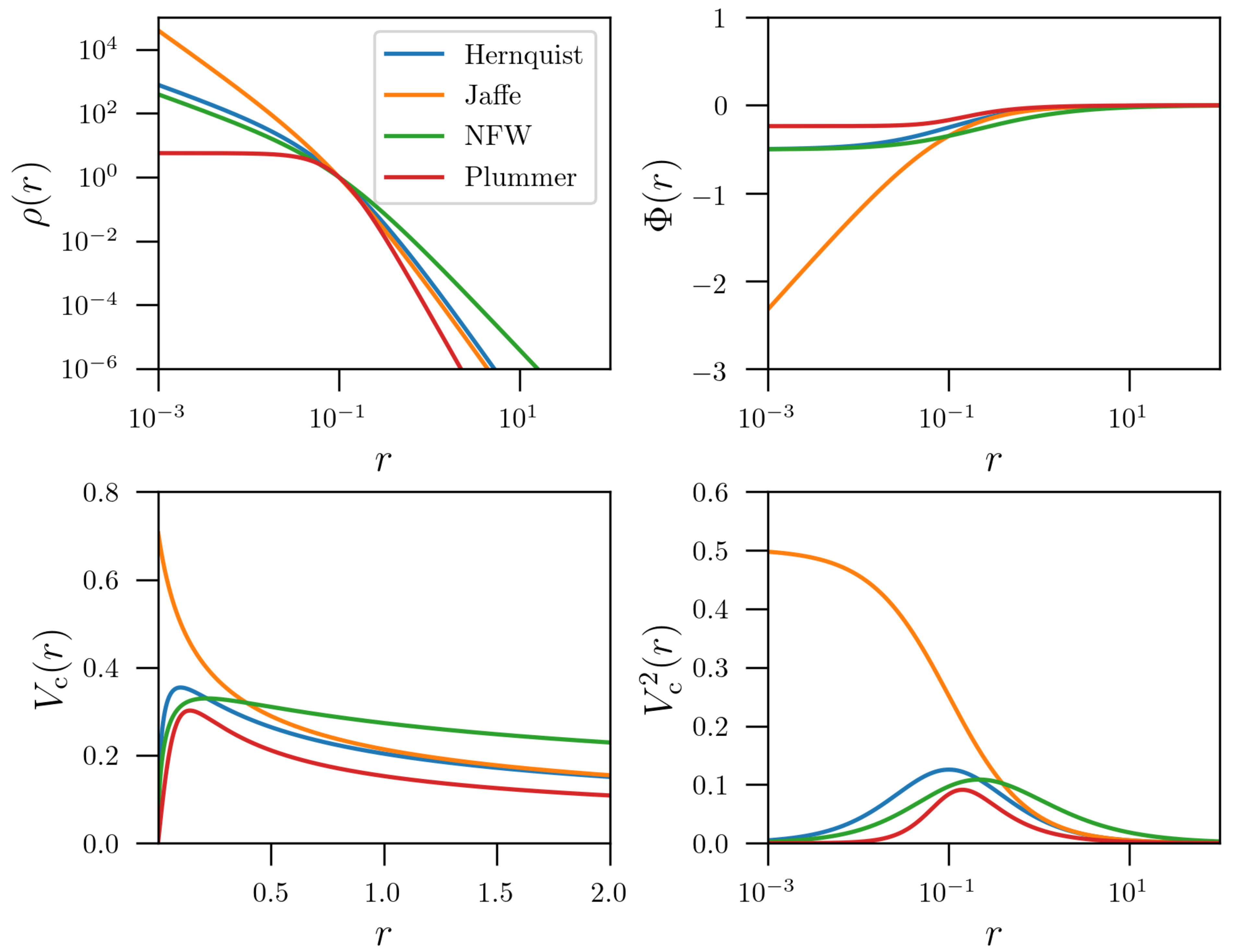
- globular clusters
- bulges, elliptic. gal.
- elliptic. galaxies
- dark haloes

# Generic two power density model

$$M(r) = 4\pi\rho_0 a^3 \times \begin{cases} \frac{r/a}{1+r/a} & \text{(Jaffe)} \\ \frac{(r/a)^2}{2(1+r/a)^2} & \text{(Hernquist)} \\ \ln(1+r/a) - \frac{r/a}{1+r/a} & \text{(NFW)} \end{cases}$$

- diverges !!

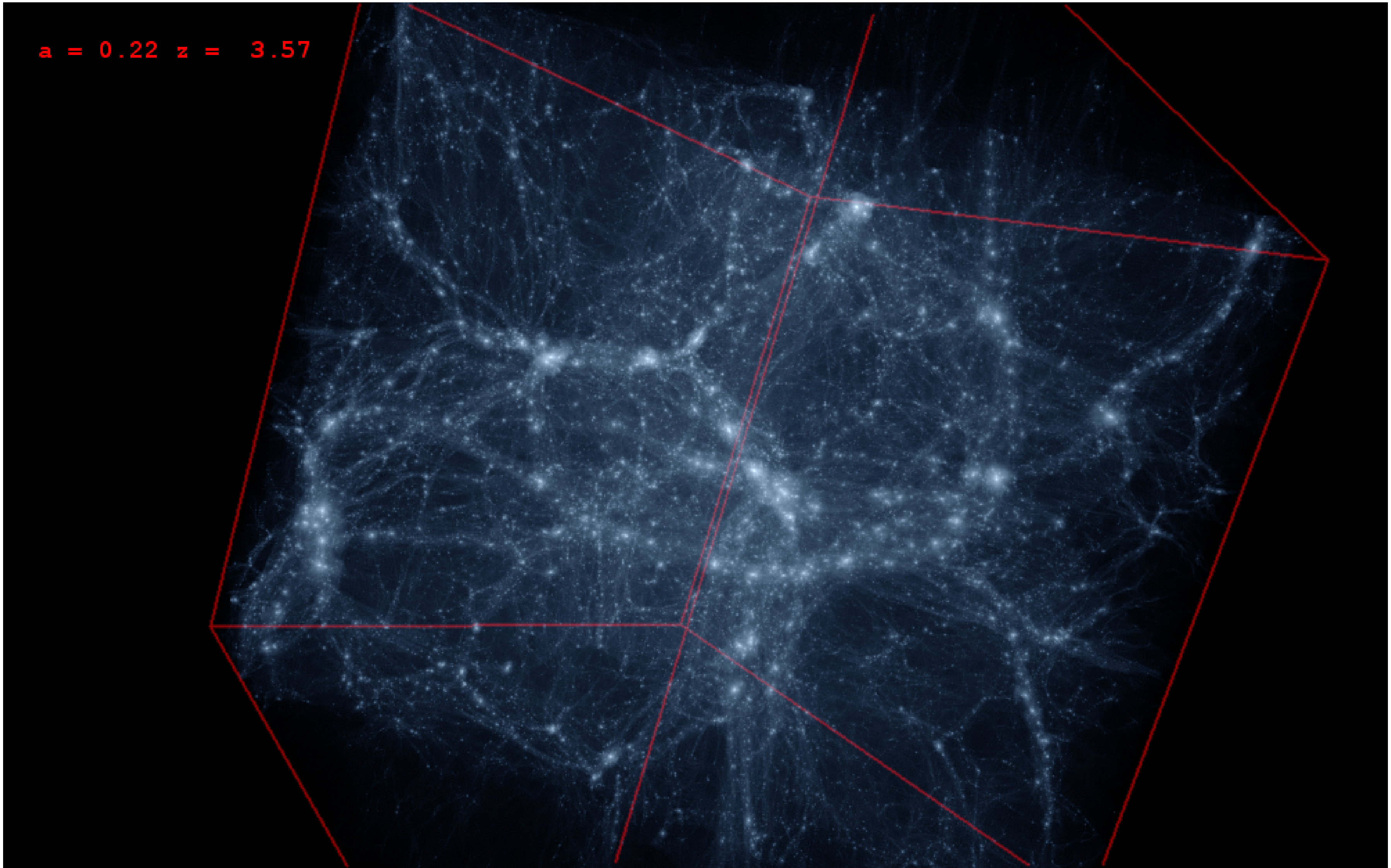
$$\Phi(r) = -4\pi G\rho_0 a^2 \times \begin{cases} \ln(1+a/r) & \text{(Jaffe)} \\ \frac{1}{2(1+r/a)} & \text{(Hernquist)} \\ \frac{\ln(1+r/a)}{r/a} & \text{(NFW)} \end{cases}$$



Hernquist
Jaffe
NFW
Plummer
$\rho(r)$
$\Phi(r)$
$V_c(r)$
$V_c^2(r)$
$r$
$10^4$
$10^2$
$10^0$
$10^{-2}$
$10^{-4}$
$10^{-6}$
$10^{-3}$
$10^{-1}$
$10^1$
1
0
−1
−2
−3
0.0
0.2
0.4
0.6
0.8
0.5
1.0
1.5
2.0
0.1
0.3
0.5

# NFW (Navarro, Frenk & White 1995, 1996)

- Density profile that fit dark matter haloes formed in LCDM numerical simulations



# NFW (Navarro, Frenk & White 1995, 1996)

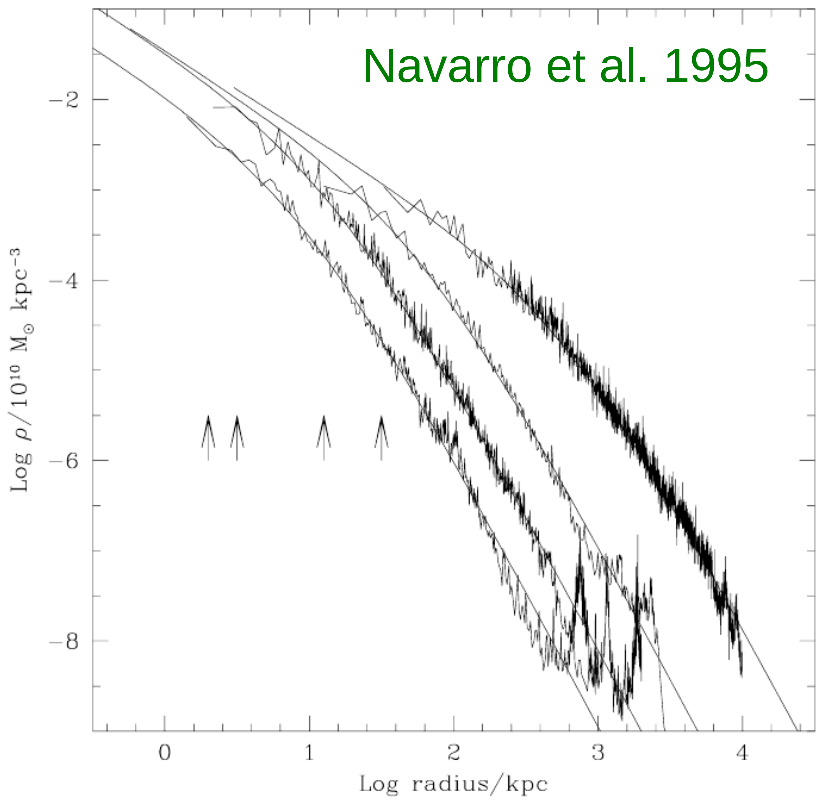



Fig. 3.— Density profiles of four halos spanning four orders of magnitude in mass. The arrows indicate the gravitational softening, $h_g$, of each simulation. Also shown are fits from eq.3. The fits are good over two decades in radius, approximately from $h_g$ out to the virial radius of each system.

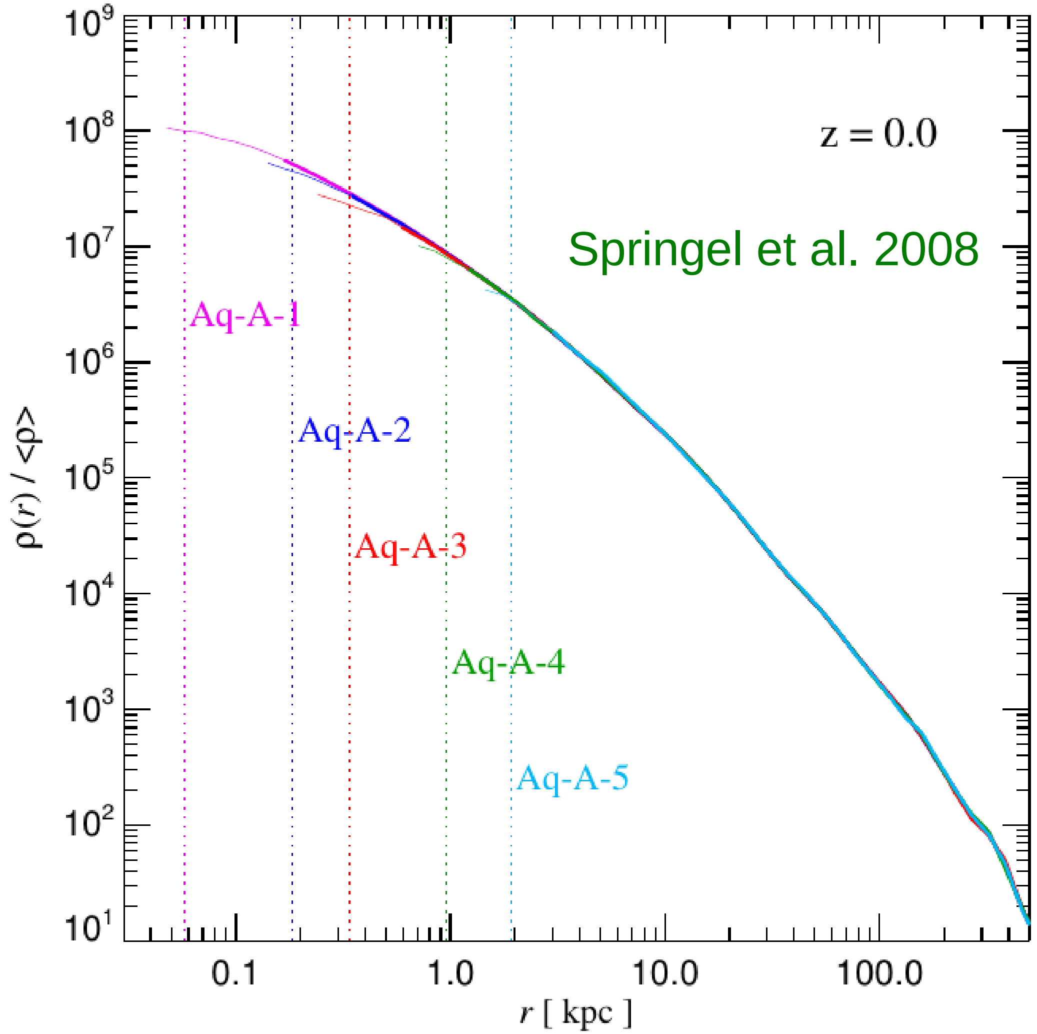
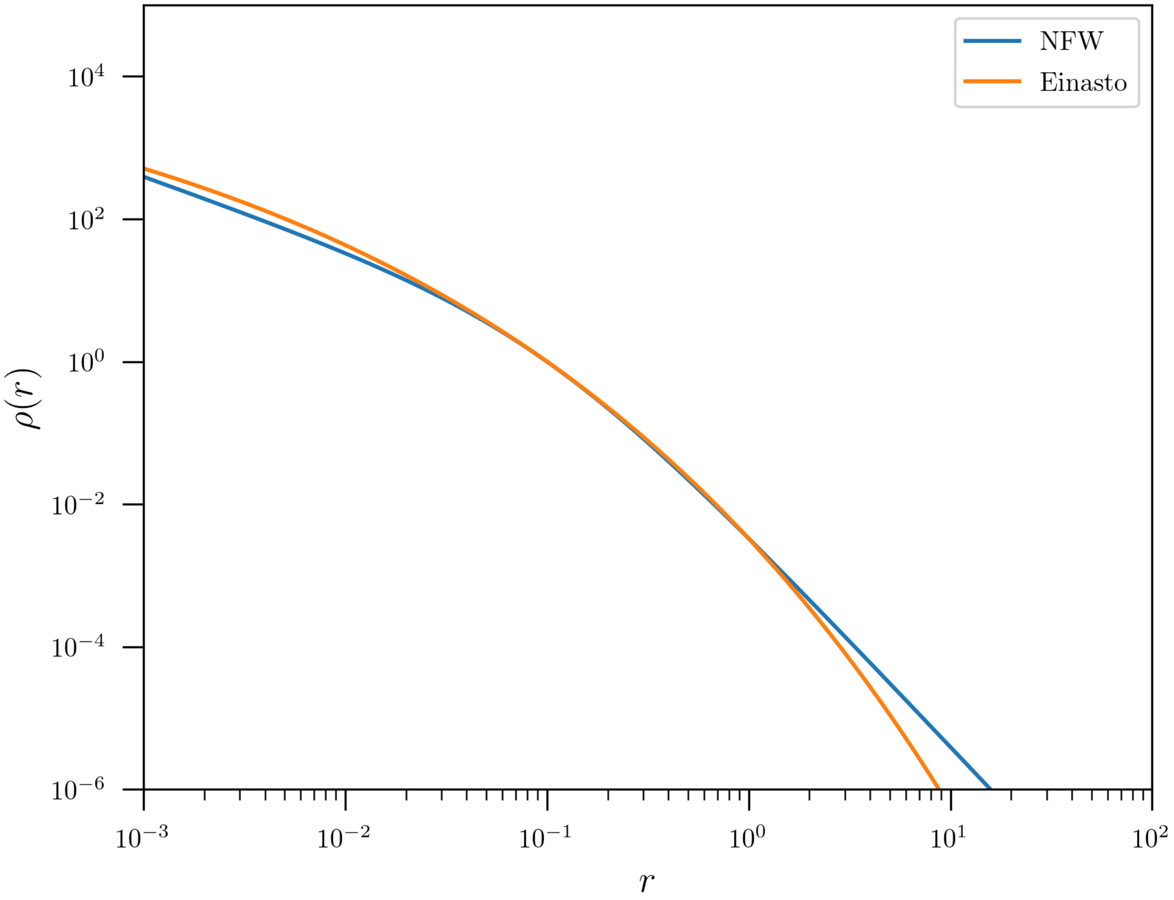



**Figure 4.** Spherically averaged density profile of the Aq-A halo at $z = 0$, at different numerical resolutions. Each of the pro-

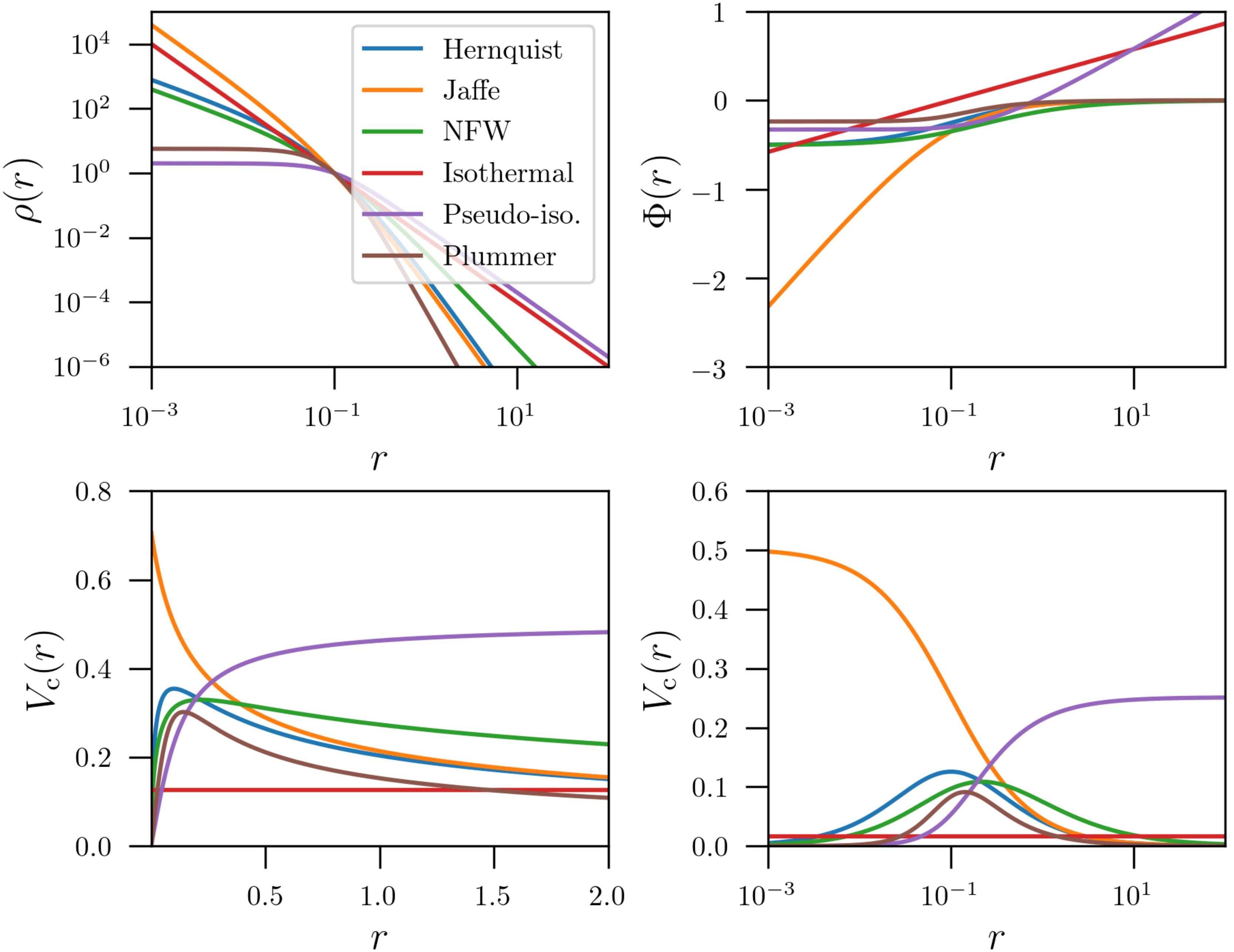
# Einasto model

$$\rho(r) = \rho_0 \exp\left[-(r/a)^{1/m}\right] \quad (m \cong 6)$$




- Alternative to NFW

# Spherical systems model comparison



# Axisymmetric models for disk galaxies

$$\rho(\vec{x}) = \rho(R, |z|)$$

$$R = \sqrt{x^2 + y^2}$$

# Examples of axisymmetric models

# “Potential based” models

# Kuzmin disk

Kuzmin 1956

$$\Phi_{\mathrm{K}}(R,z) = -\frac{GM}{\sqrt{R^2 + (a + |z|)^2}} = -\frac{GM}{\sqrt{R^2 + z^2 + a^2 + 2\,a\,|z|}}$$

Comparison with Plummer:

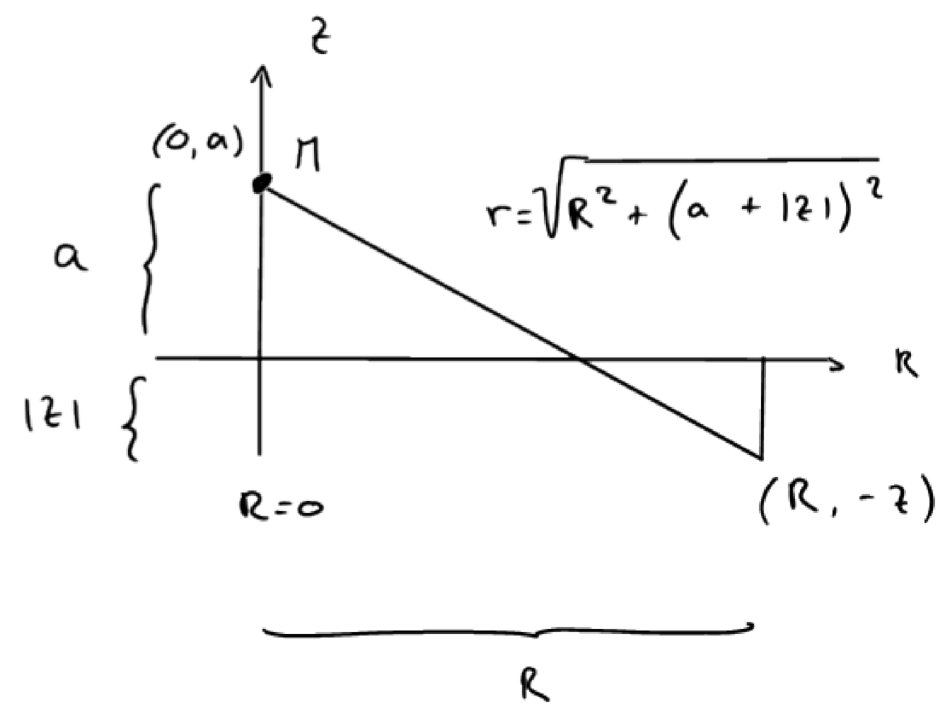
$$\Phi_{\mathrm{P}}(R,z) = -\frac{GM}{\sqrt{R^2 + z^2 + a^2}}$$

## Equivalent to the following configuration

Potential due to

a mass $M$ at $(0, a)$

$$\Rightarrow \quad -\frac{GM}{r} = -\frac{GM}{\sqrt{R^2 + (a + |z|)^2}}$$



# Kuzmin disk

Kuzmin 1956

$$\Phi_{\rm K}(R,z) = -\frac{GM}{\sqrt{R^2+(a+|z|)^2}}$$

Plummer based model

$$\Sigma_{\rm K}(R) = \frac{aM}{2\pi(R^2+a^2)^{3/2}}$$


Infinitely thin disk

$$V_{c,{\rm K}}^2(R) = \frac{G\,M\,R^2}{\left(R^2+a^2\right)^{3/2}}$$

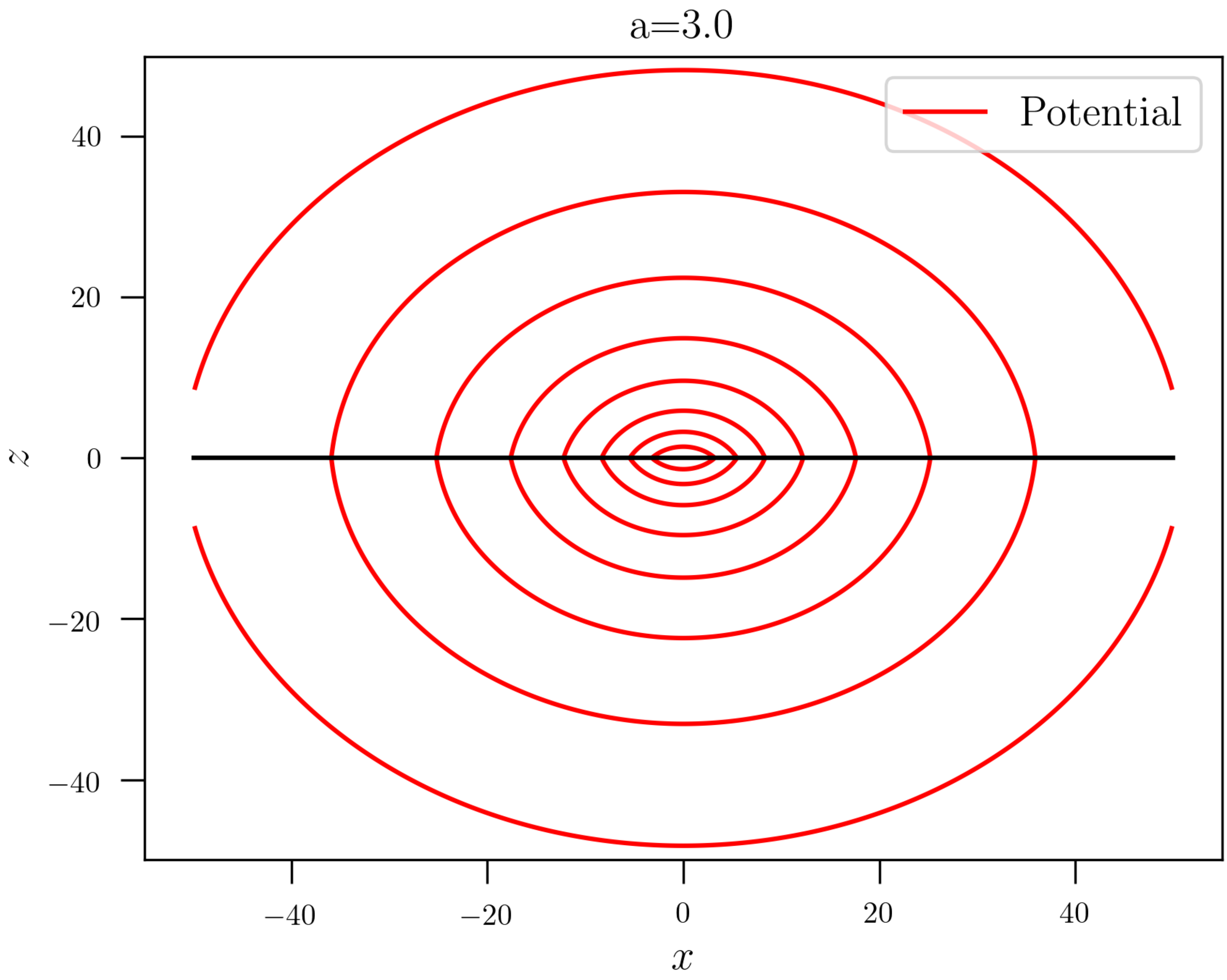
Equivalent to the Plummer model

$$V_{c,{\rm P}}^2(r) = \frac{GMr^2}{(r^2+b^2)^{3/2}}$$

Note: for an axi-symmetric model, the circular velocity is computed in the plane z=0.

$$V_c^2(R) = \frac{1}{R}\frac{\mathrm{d}\Phi(\mathrm{R},\mathrm{z}=0)}{\mathrm{dR}}$$

# Kuzmin disk



# Miyamoto-Nagai potential

Miyamoto & Nagai 1975

$$\Phi_{\rm MN}(R,z) = -\frac{GM}{\sqrt{R^2 + (a+\sqrt{z^2+b^2})^2}}$$

b=0 → Kuzmin

$$\rho_{\rm MN}(R,z) = \left(\frac{b^2 M}{4\pi}\right) \frac{aR^2 + (a+3\sqrt{z^2+b^2})(a+\sqrt{z^2+b^2})^2}{[R^2+(a+\sqrt{z^2+b^2})^2]^{5/2}(z^2+b^2)^{3/2}}$$

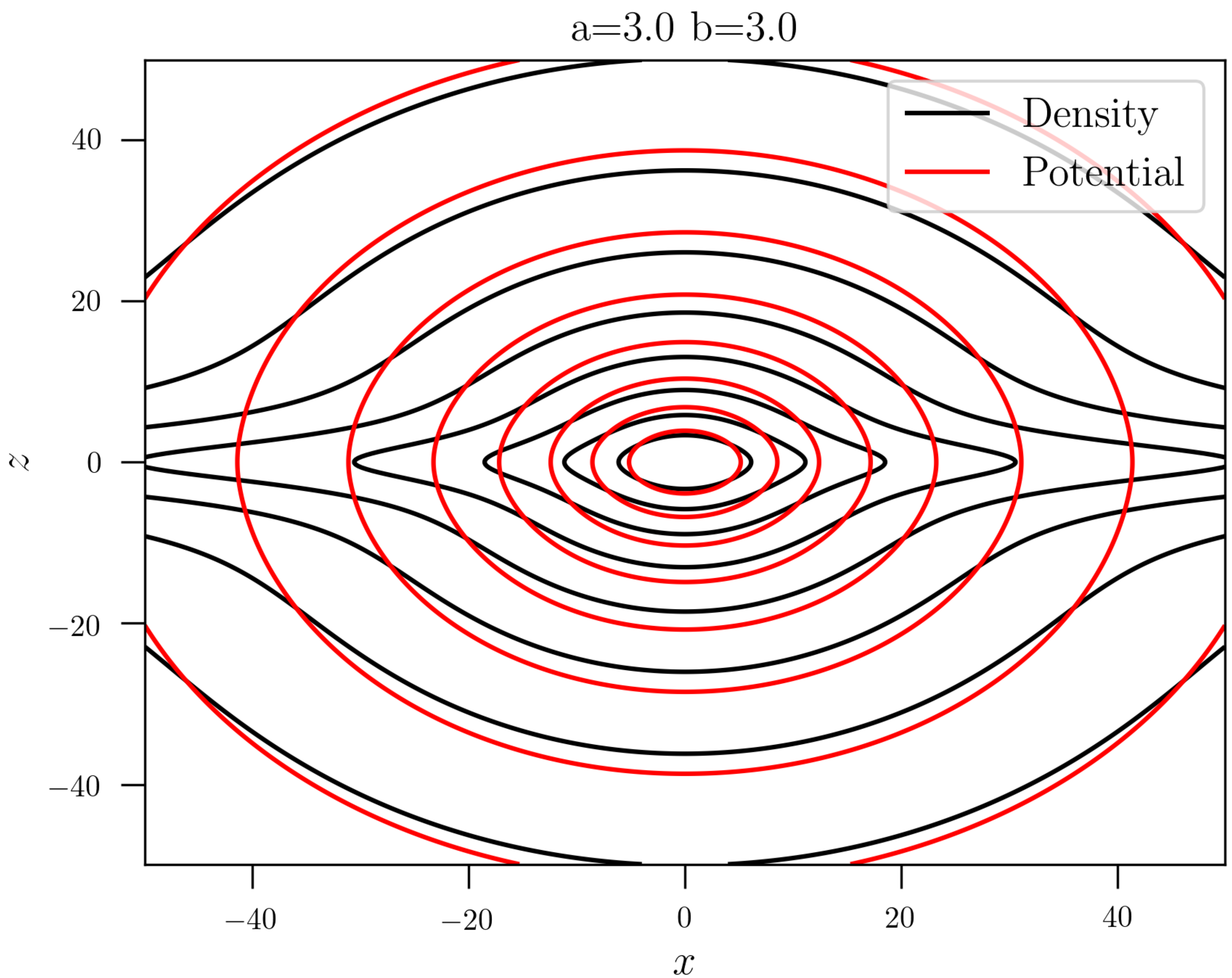
$$V^2_{c,\rm MN}(R) = \frac{G\,M\,R^2}{\left(R^2+(a+b)^2\right)^{3/2}}$$

Equivalent to the Plummer model

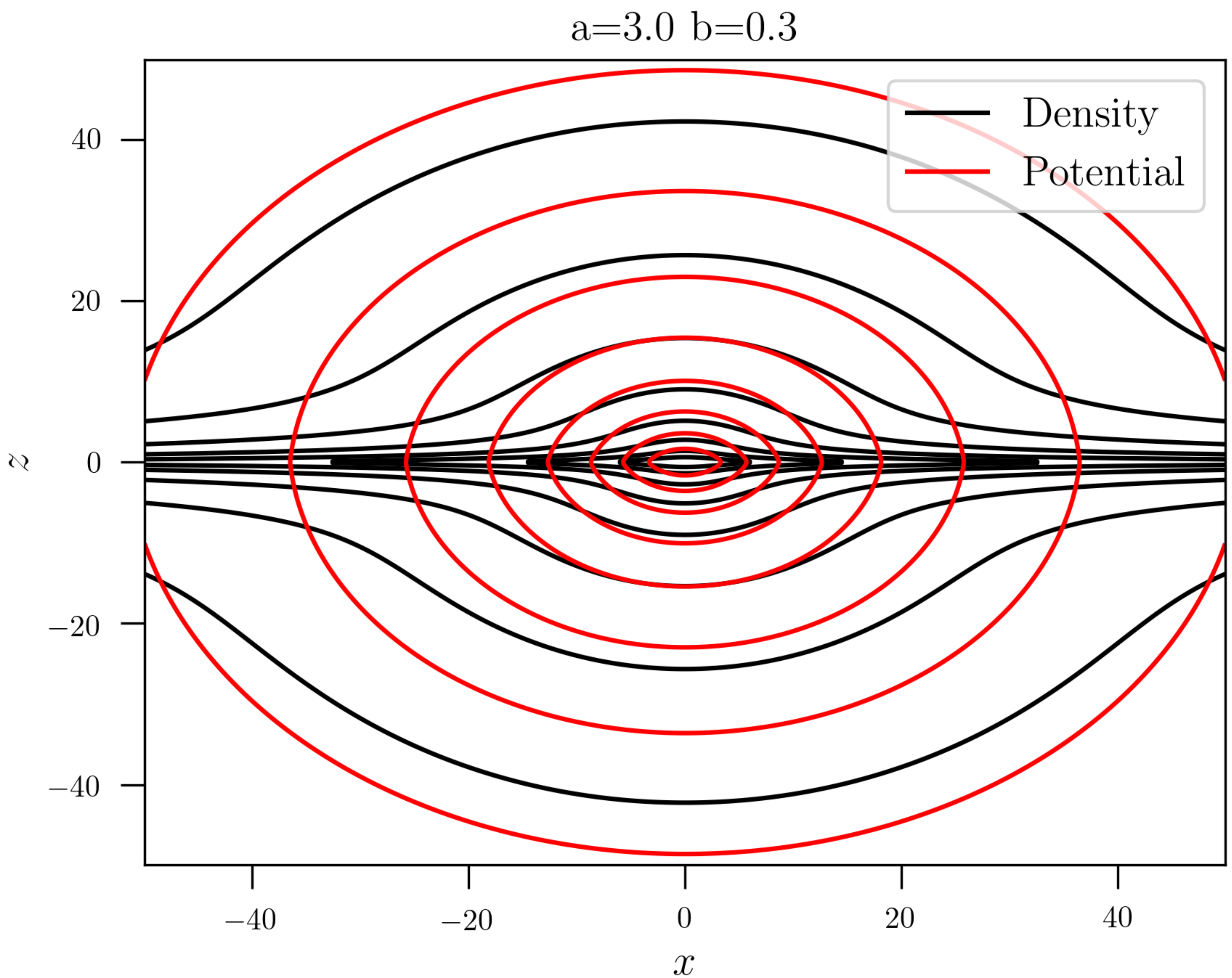
$$V^2_{c,\rm P}(r) = \frac{GMr^2}{(r^2+b^2)^{3/2}}$$


Better parametrisation :
Revaz & Pfenniger 2004

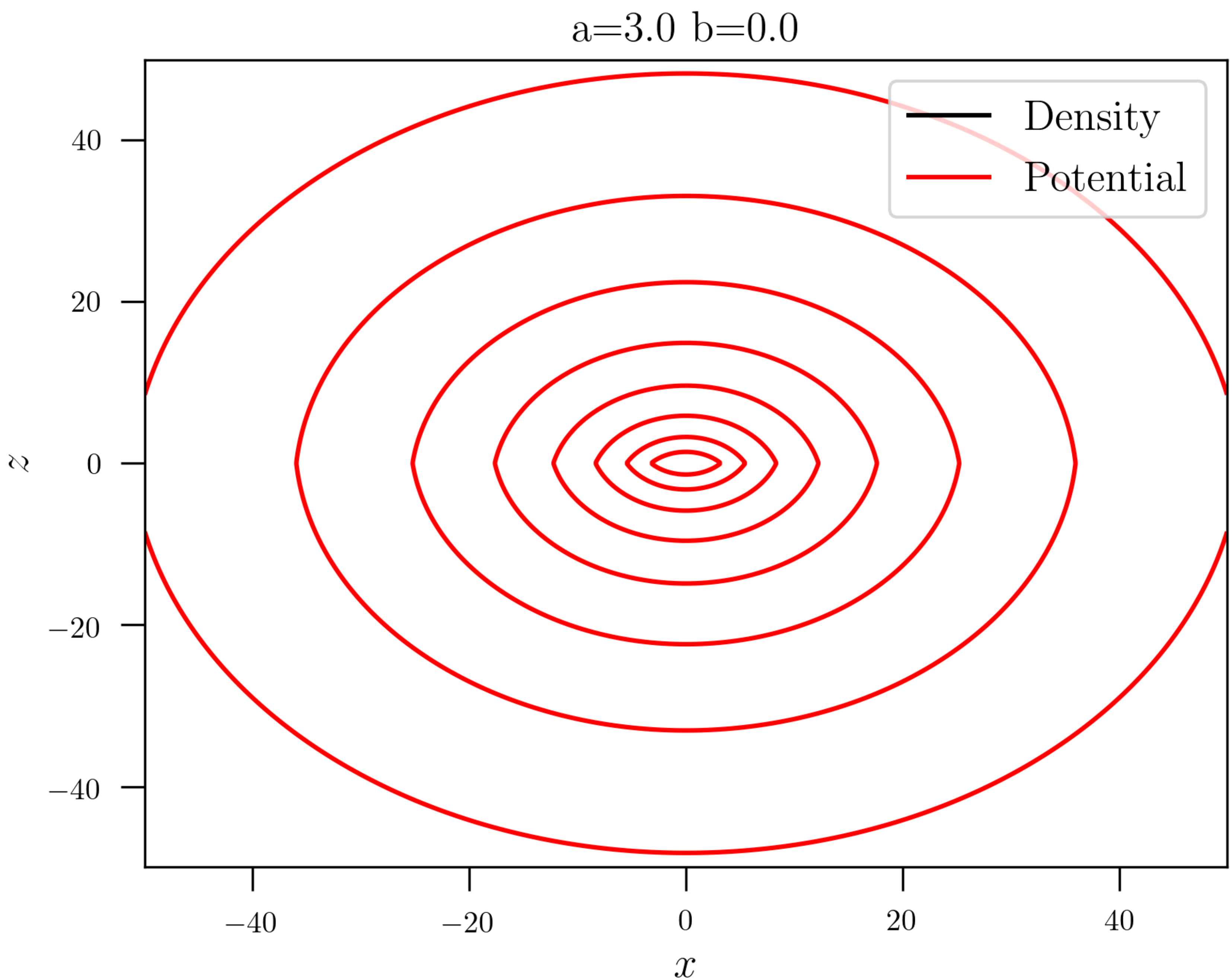
# Miyamoto-Nagai potential



# Miyamoto-Nagai potential



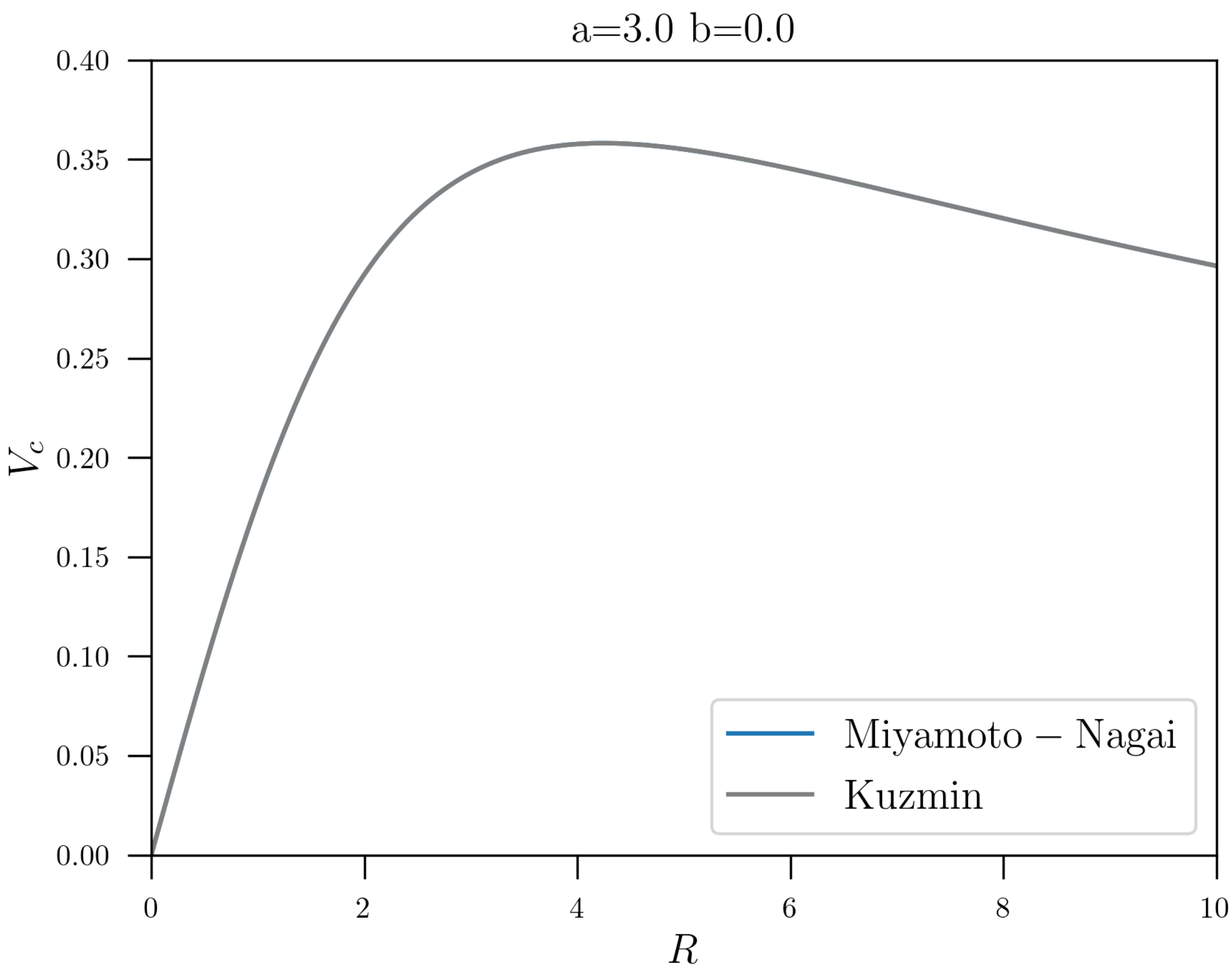
# Miyamoto-Nagai potential



# Miyamoto-Nagai potential

Miyamoto & Nagai 1975

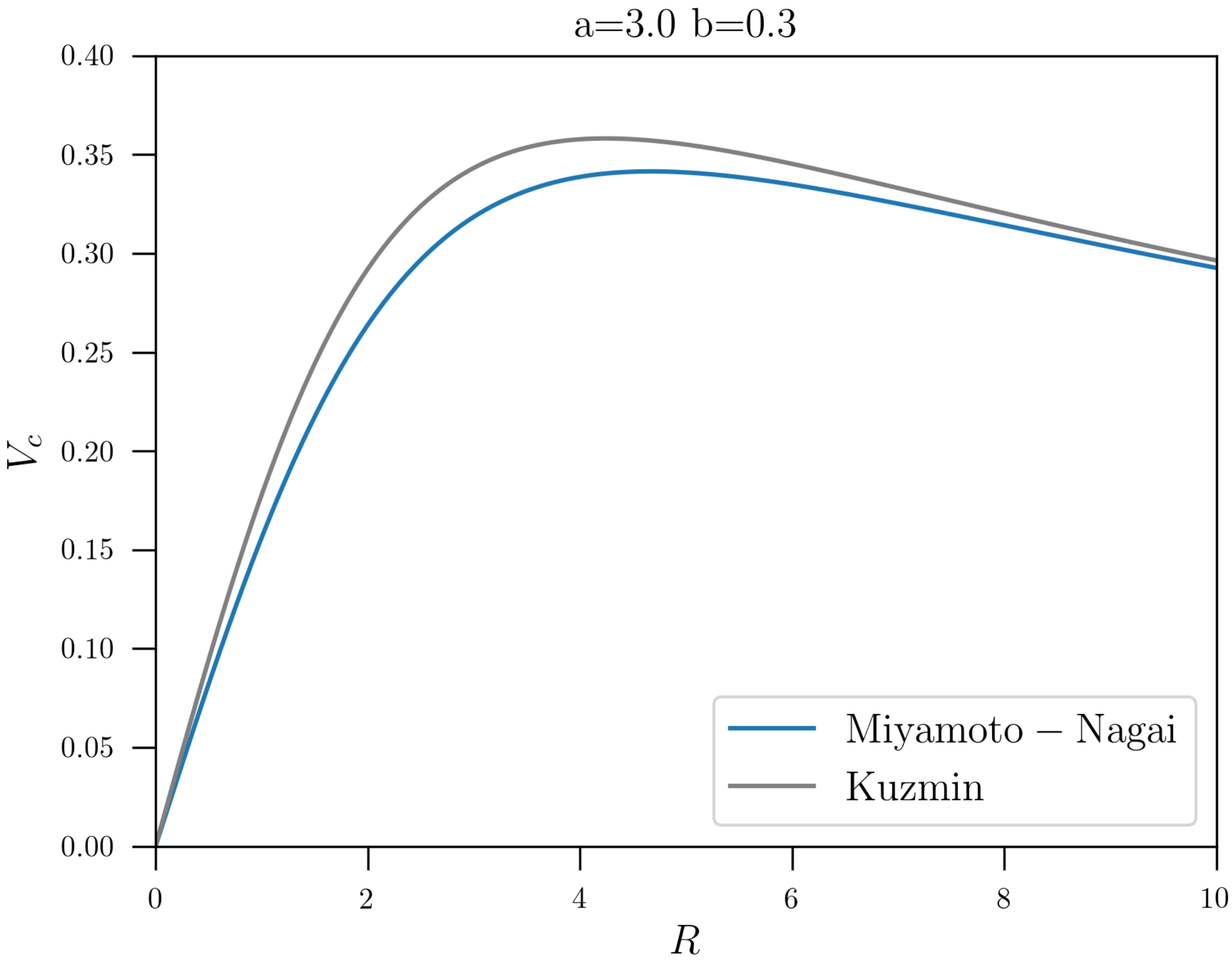
## Circular velocity rotation curve



# Miyamoto-Nagai potential

Miyamoto & Nagai 1975

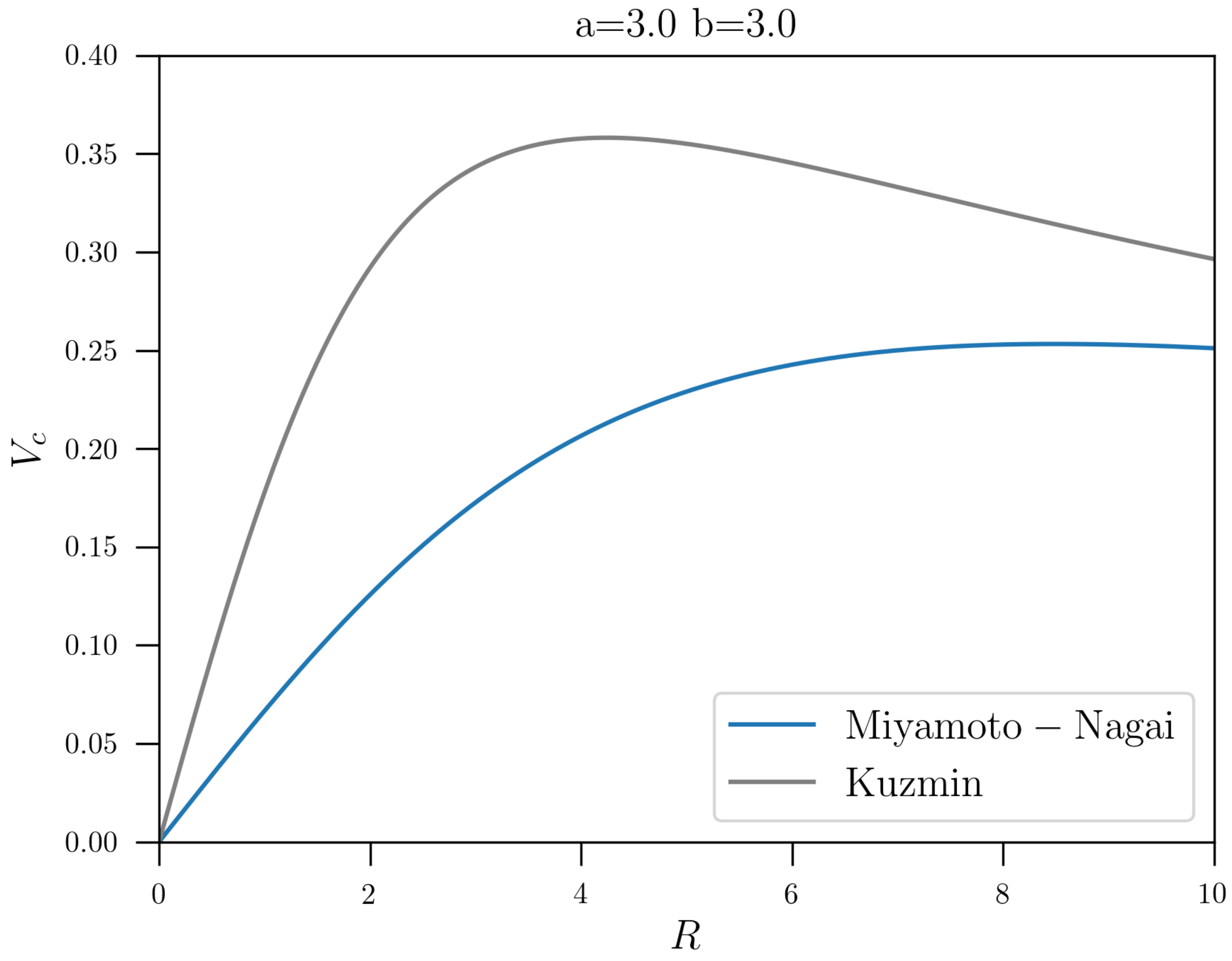
## Circular velocity rotation curve



# Miyamoto-Nagai potential

Miyamoto & Nagai 1975

## Circular velocity rotation curve



# Logarithmic potential

$$\Phi_{\log}(R,z) = \frac{1}{2}V_0^2 \ln\left(R_c^2 + R^2 + \frac{z^2}{q^2}\right)$$

Rc=0 and q=1
→ Isothermal sphere

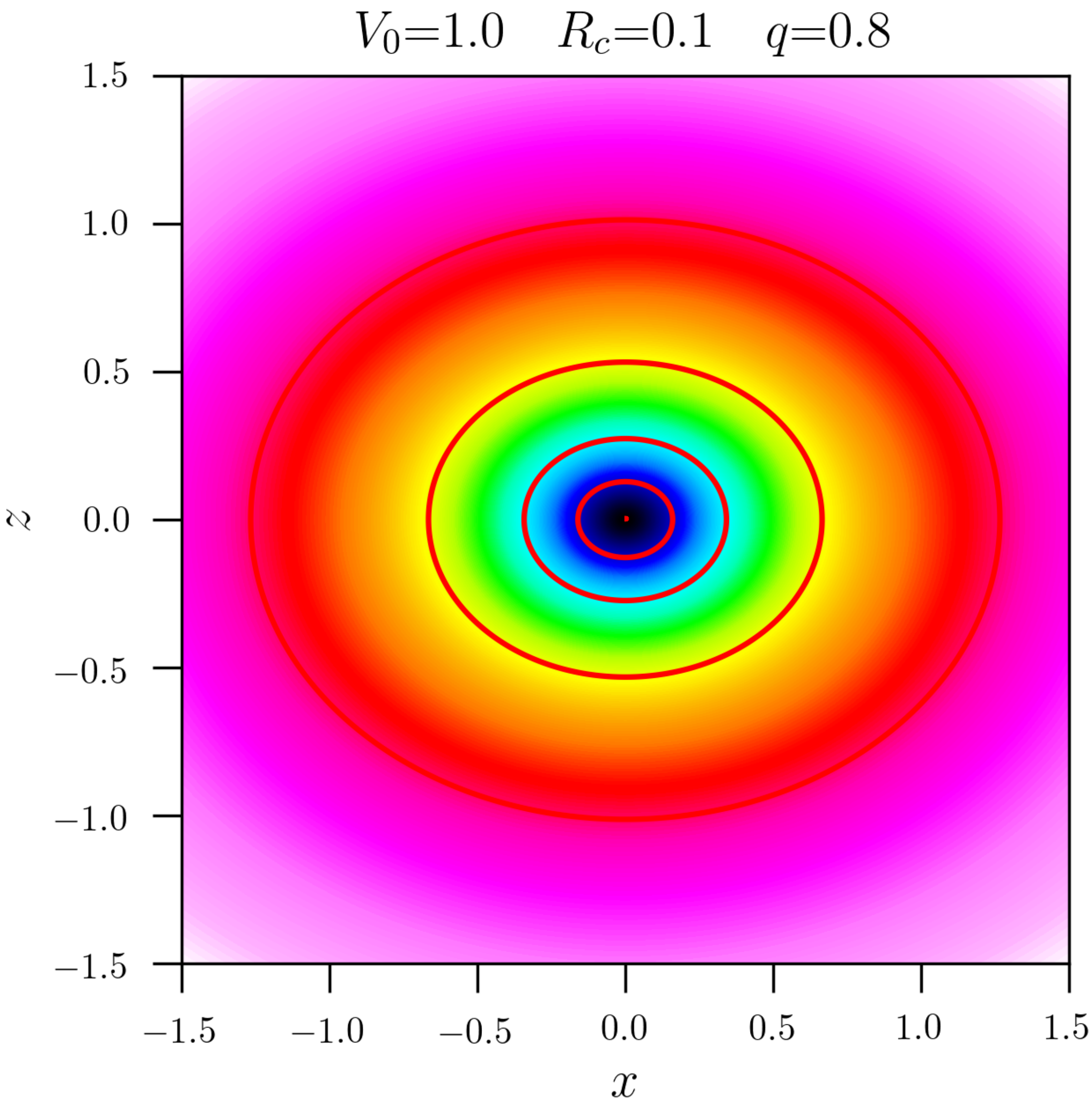
$$\rho_{\log}(R,z) = \frac{V_0^2}{4\pi G q^2}\frac{(2q^2+1)R_c^2 + R^2 + (2-1/q^2)z^2}{\left(R_c^2 + R^2 + (z^2/q^2)\right)^2}$$

- negative for $q < 1\sqrt(2) \cong 0.707$

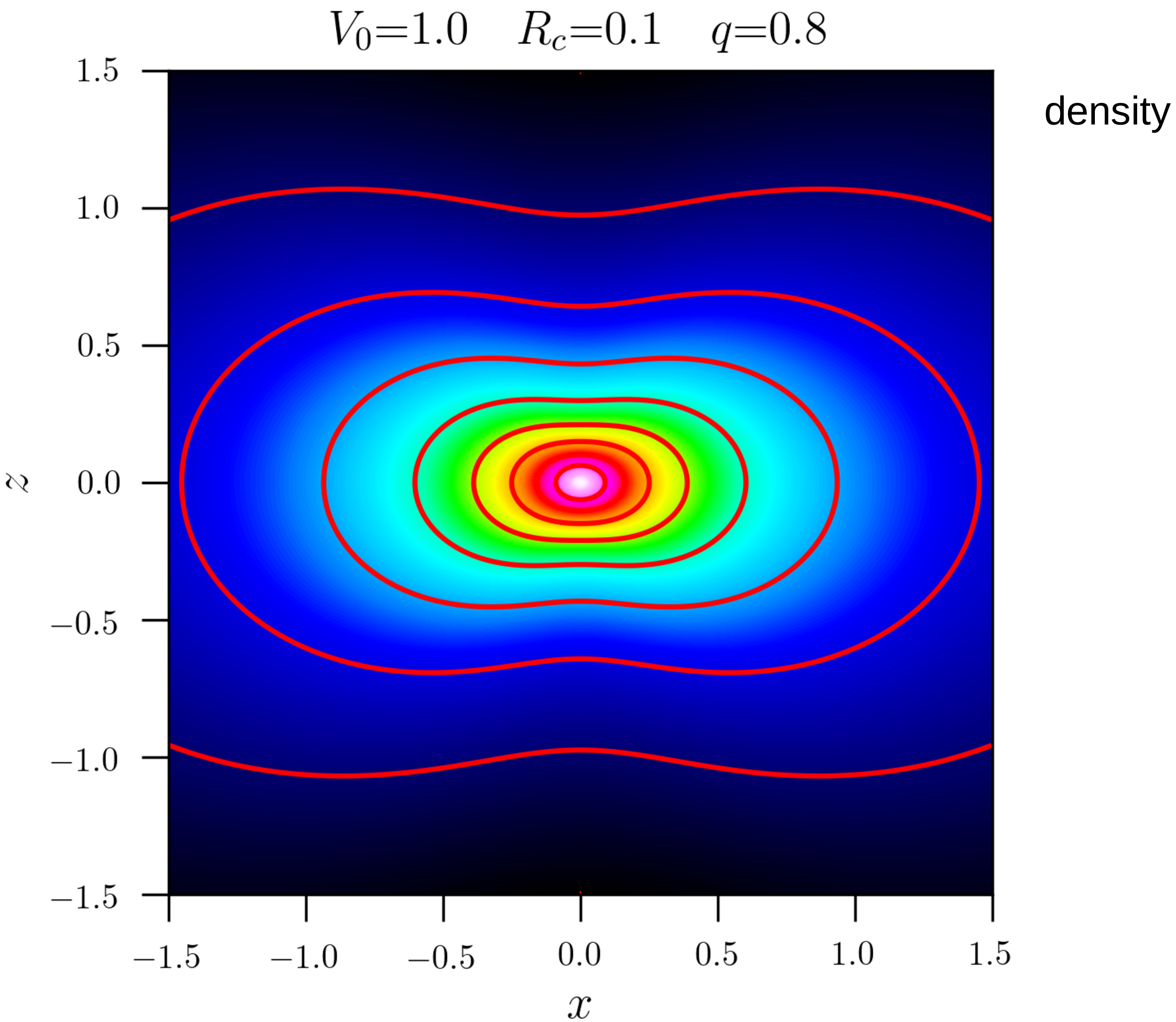
$$V_{c,\log}^2(R) = V_0^2 \frac{R^2}{R_c^2 + R^2}$$

- does not depends on q
- flat rotation curve at large radius

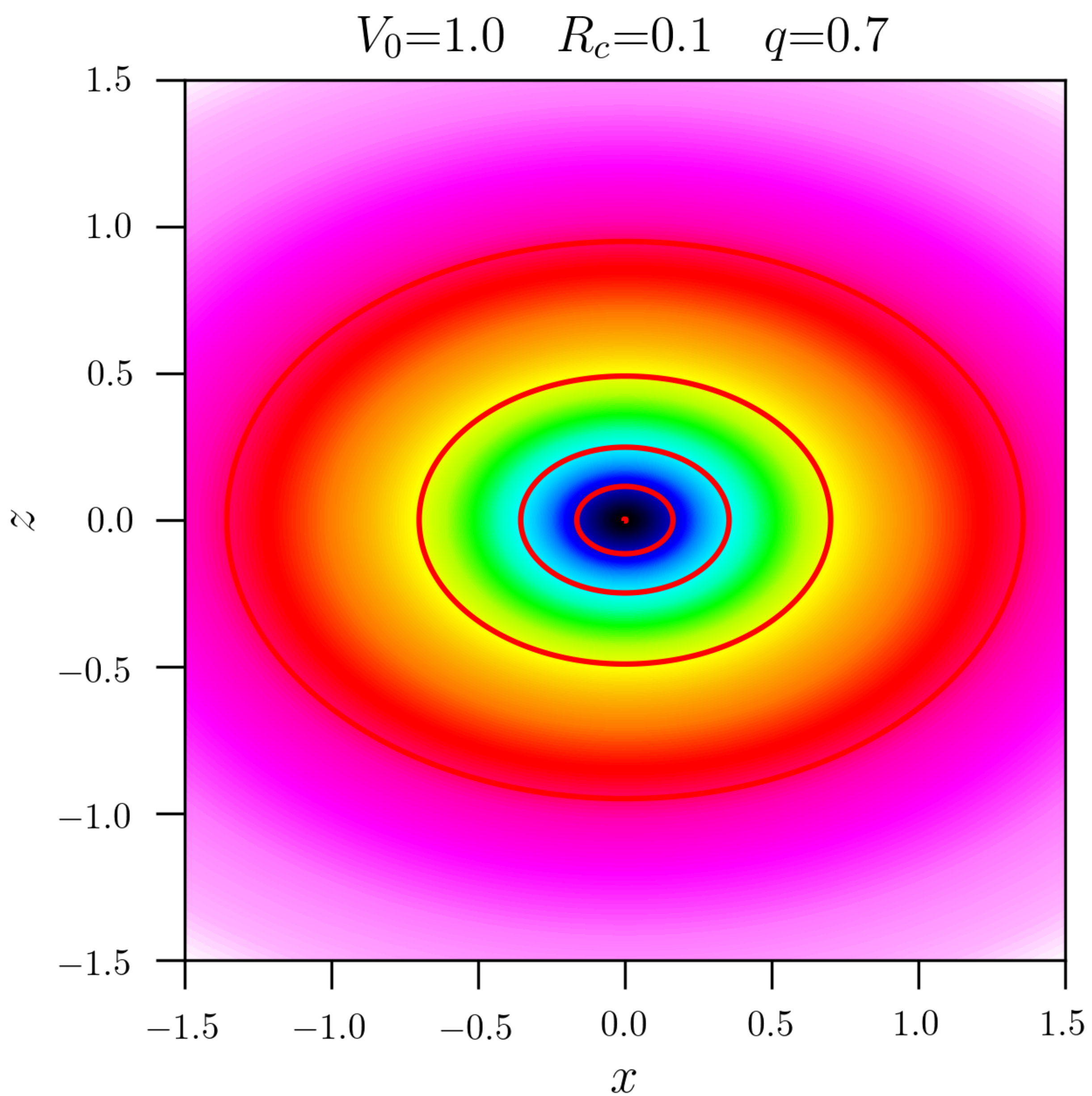
# Logarithmic potential



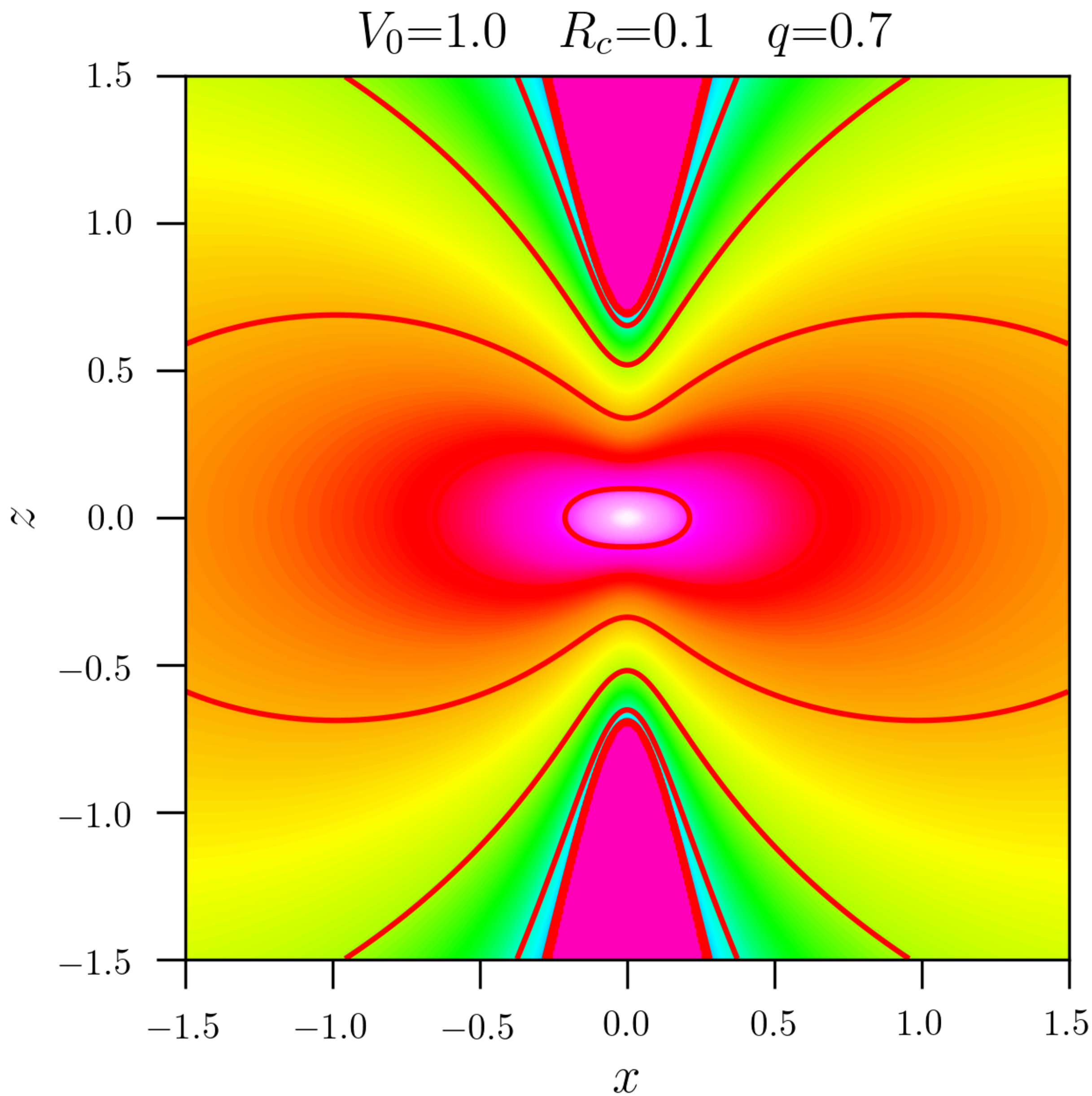
# Logarithmic potential



# Logarithmic potential

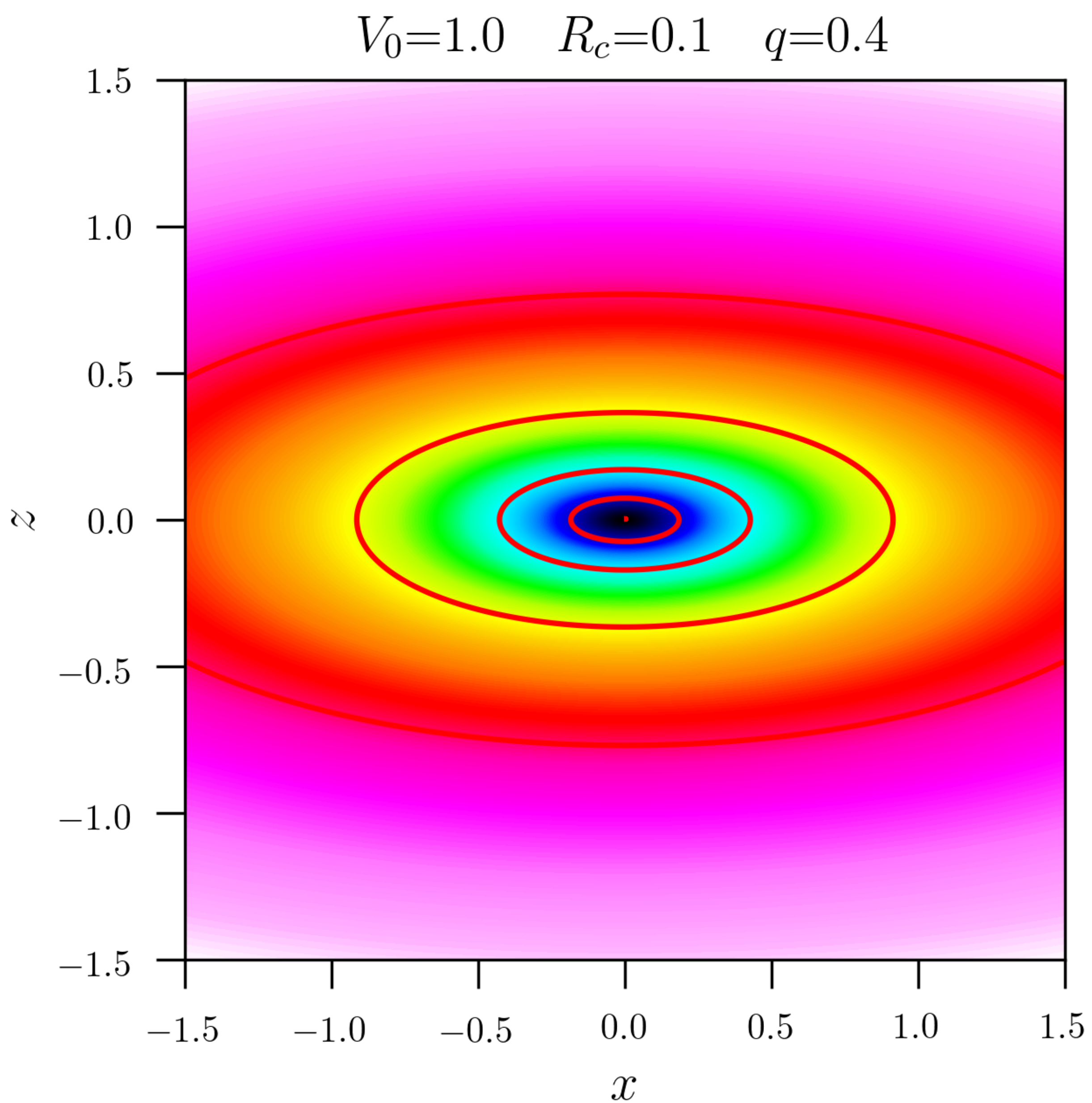


# Logarithmic potential

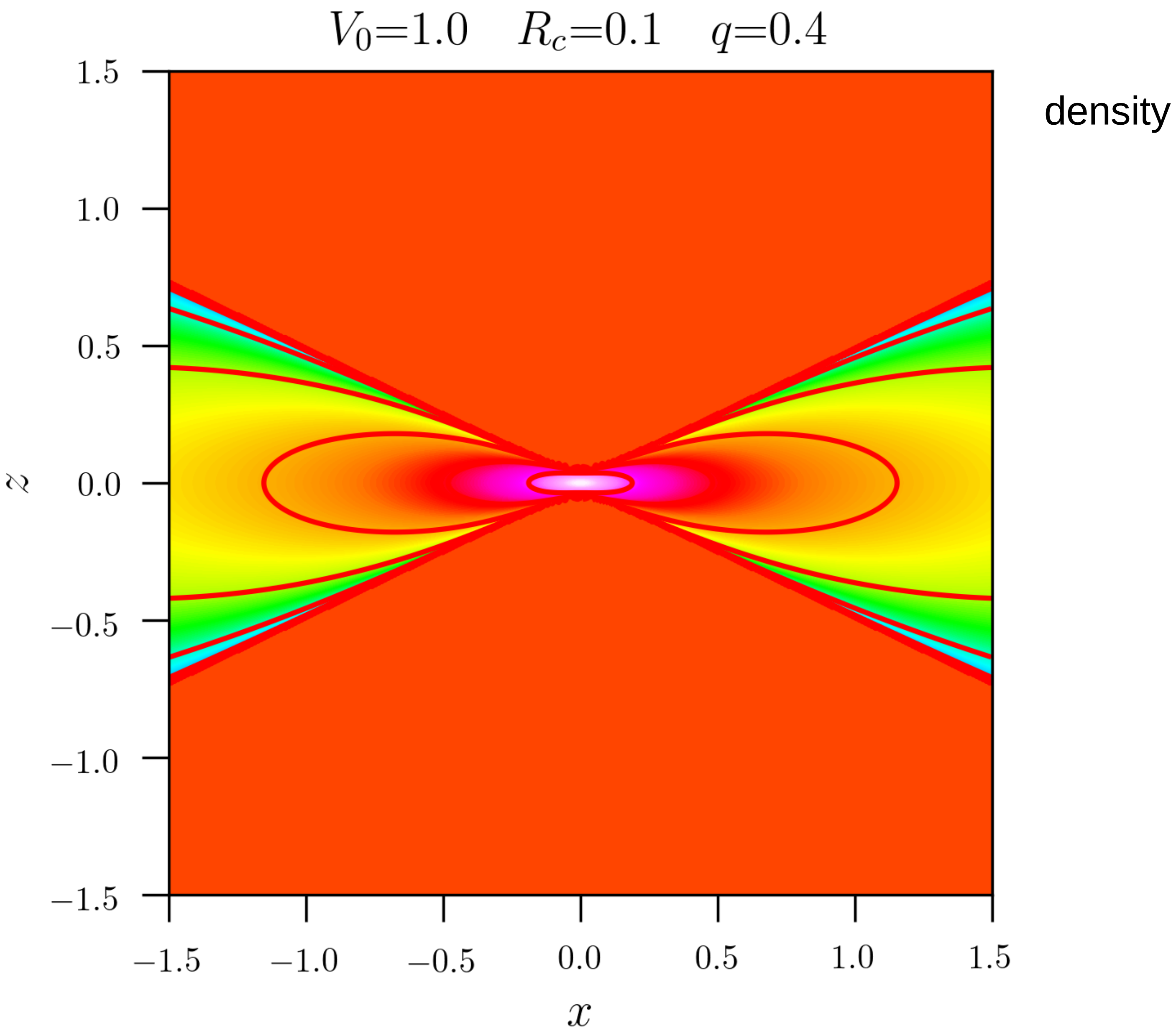



density

# Logarithmic potential

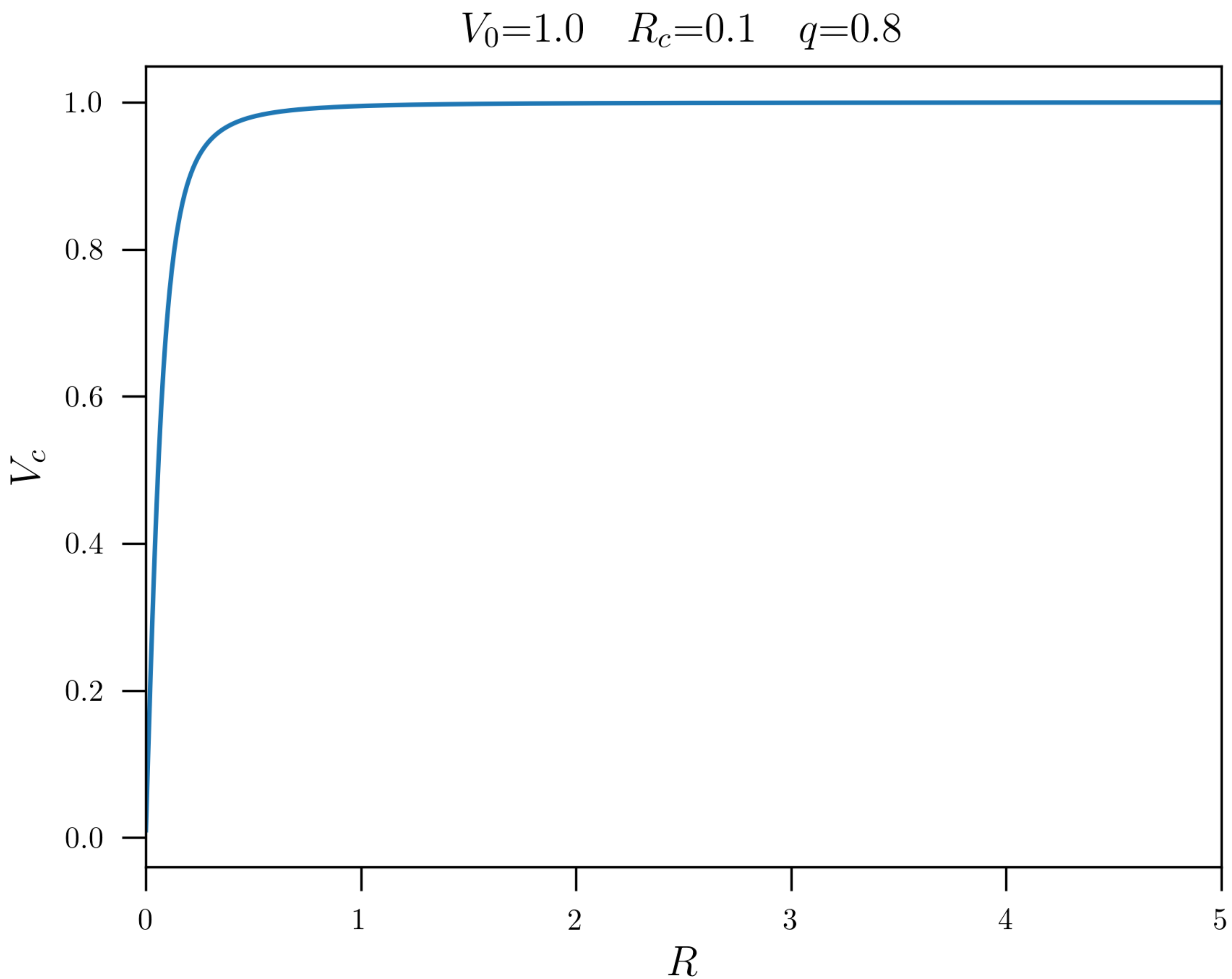


# Logarithmic potential



# Logarithmic potential

## Circular velocity rotation curve



# The End