

The gravity force and Newtonian mechanics

Outlines

Newtonian Mechanics:

- refreshing memory

The gravity : a long distance force

- collision-less systems
- the relaxation time

Potential Theory : general results

- Gravitational field force, gravitational potential
- Gauss Law
- Poisson Equation
- Total potential energy

Refreshing memory...

Newtonian mechanics

Newtonian mechanics : a very short remainder

point mass : mass m

position \vec{x}

velocity $\vec{v} = \frac{d\vec{x}}{dt}$

momentum $\vec{p} = m \vec{v} = m \frac{d\vec{x}}{dt}$

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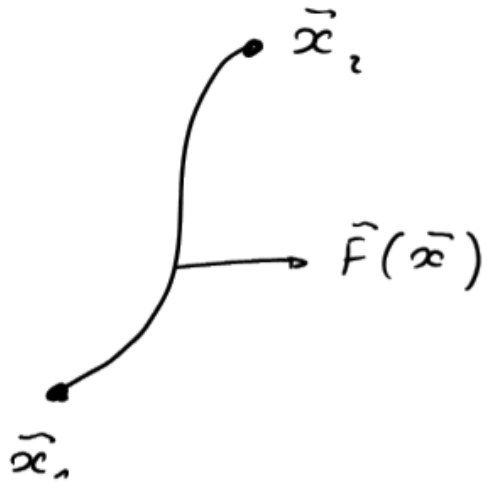
Newton second law

$$\frac{d}{dt}(\vec{p}) = m \frac{d^2\vec{x}}{dt^2} = \vec{F}$$

• \vec{F} : a force

\vec{p} is constant in absence of a force

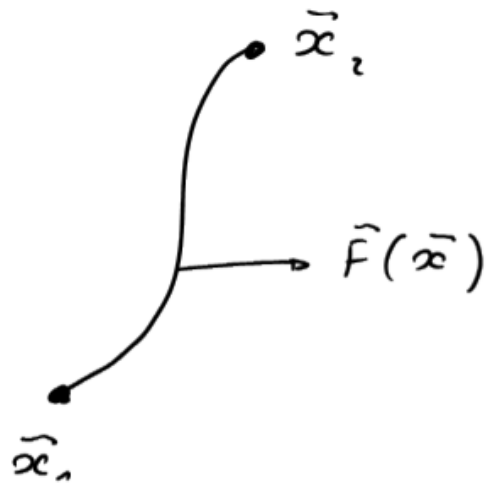
Work : work done by a force in moving the particle
from \vec{x}_1 to \vec{x}_2



$$W_{12} = - \int_{\vec{x}_1}^{\vec{x}_2} \vec{F}(\vec{x}) \cdot d\vec{x}$$

$$[W_{12}] = 9 \frac{\text{cm}^2}{\text{s}^2} \\ = \text{erg}$$

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Power of a force (energy rate)

$$\frac{\text{erg}}{\text{s}}$$

$$\text{with } \frac{\partial \tilde{F}(\vec{x})}{\partial x_i} = F_i(\vec{x})$$

$$\frac{d}{dt} W_{12}(x(t)) = - \frac{d}{dt} \int_{\vec{x}_1}^{\vec{x}(t)} \vec{F}(\vec{x}) \cdot d\vec{x} = - \frac{d}{dt} \left(\tilde{F}(\vec{x}(t)) - \tilde{F}(\vec{x}_1) \right)$$

$$= - \vec{\nabla}_{\vec{x}} \tilde{F}(\vec{x}) \cdot \frac{d}{dt} (\vec{x}(t)) = - \vec{F}(\vec{x}) \vec{v}(\vec{x})$$

Kinetic energy

$$W_{12} = - \int_{\vec{x}_1}^{\vec{x}_2} \vec{F}(\vec{x}) \cdot d\vec{x}$$

integration by part gives

$$= -m \left[\vec{v}^2 \Big|_{\vec{x}_1}^{\vec{x}_2} - \int_{\vec{x}_1}^{\vec{x}_2} \vec{v} \frac{d\vec{v}}{dt} dt \right] = -m \vec{v}_2^2 + m \vec{v}_1^2 + m \underbrace{\int_{\vec{x}_1}^{\vec{x}_2} \vec{v} \frac{d\vec{v}}{dt} dt}_{-W_{12}}$$

Thus $W_{12} = \frac{1}{2} m \vec{v}_1^2 - \frac{1}{2} m \vec{v}_2^2$

$$W_{12} = K_1 - K_2$$

Newton
2nd law

$$= -m \int_{\vec{x}_1}^{\vec{x}_2} \frac{d\vec{v}}{dt} \frac{d\vec{x}}{dt} dt$$
$$= -m \int_{\vec{x}_1}^{\vec{x}_2} \frac{d\vec{v}}{dt} \cdot \vec{v} \cdot dt$$

$$\vec{x} = \vec{x}(t)$$
$$d\vec{x} = \frac{d\vec{x}}{dt} dt$$

$$K = \frac{1}{2} m \vec{v}^2 : \text{Kinetic energy}$$

Potential energy and conservative forces

A force $\vec{F}(\vec{x})$ is called conservative if the work done by this force in moving the particle from \vec{x}_1 to \vec{x}_2 is independent of the path.

Then, for any given point \vec{x}_0

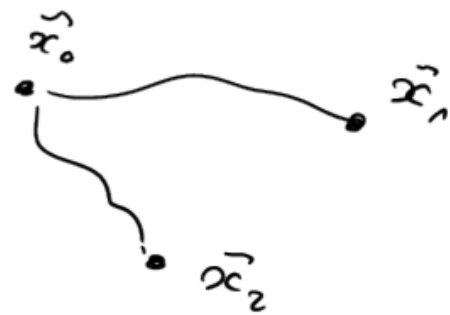
we can define the function (potential) $V_0(\vec{x})$

$$V_0(\vec{x}) := V_{0\vec{x}} = - \int_{\vec{x}_0}^{\vec{x}} \vec{F}(\vec{x}') d\vec{x}'$$

Then $W_{12} = W_{10} + W_{02}$

$W_{12} = V(\vec{x}_2) - V(\vec{x}_1)$

Useful convention : $\vec{x}_0 \rightarrow \infty$ (far away from all interacting bodies)



Gradient of the potential

$$\vec{\nabla}_{\vec{x}} \cdot V(\vec{x}) = -\vec{\nabla}_{\vec{x}} \cdot \left[\int_{\vec{x}_0}^{\vec{x}} \vec{F}(\vec{x}') d\vec{x}' \right] = -\vec{\nabla}_{\vec{x}} \left(\underbrace{F(\vec{x}) - F(\vec{x}_0)}_{\text{ck} \Rightarrow 0} \right) = -\vec{F}(\vec{x})$$

with $\frac{\partial F(\vec{x})}{\partial x_i} = F_i(\vec{x})$

$$\vec{\nabla}_{\vec{x}} \cdot V(\vec{x}) = -\vec{F}(\vec{x})$$

We can represent a conservative force field by its potential

Total Energy E

$$E := K + V = \frac{1}{2} m \vec{v}^2 + V(\vec{x})$$

Theorem

The energy E of a system evolving under conservative forces $\vec{F}(\vec{x})$ (associated to a potential $V(\vec{x})$) is constant.



$$E_1 = E(\vec{x}_1) = \frac{1}{2} m v_1^2 + V(\vec{x}_1)$$

$$E_2 = E(\vec{x}_2) = \frac{1}{2} m v_2^2 + V(\vec{x}_2)$$

$$E_1 - E_2 = \underbrace{K_1 - K_2}_{W_{12}} + \underbrace{V(\vec{x}_1) - V(\vec{x}_2)}_{-W_{12}}$$

$$= 0$$

#

Angular momentum and Torque

Angular momentum $\vec{L} = \vec{x} \times \vec{p}$

Torque $\vec{N} = \vec{x} \times \vec{F}$

$$\begin{aligned}\frac{d\vec{L}}{dt} &= \frac{d\vec{x}}{dt} \times \vec{p} + \vec{x} \times \frac{d\vec{p}}{dt} \\ &= \underbrace{\vec{v} \times \vec{p}}_{=0} + \underbrace{\vec{x} \times \vec{F}}_{\vec{N}}\end{aligned}$$

$$\frac{d\vec{L}}{dt} = \vec{N}$$

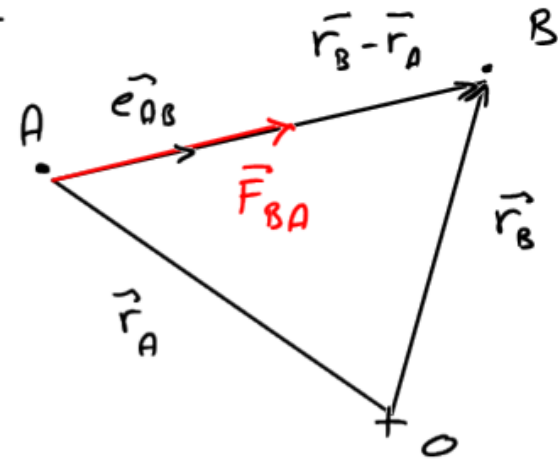
**The Gravity :
a long distance force**

The gravity : a long range force

$$\vec{F}_{BA} = \frac{G m_A m_B}{|\vec{r}_B - \vec{r}_A|^2} \vec{e}_{AB}$$

$$\vec{F}_{BA} = \frac{G m_A m_B}{|\vec{r}_B - \vec{r}_A|^3} \vec{r}_B - \vec{r}_A$$

$$|\vec{F}_{BA}| = \frac{G m_A m_B}{r_{AB}^2}$$



$$\vec{e}_{AB} = \frac{\vec{r}_B - \vec{r}_A}{|\vec{r}_B - \vec{r}_A|}$$

$$r_{AB} = |\vec{r}_B - \vec{r}_A|$$

$$[G] = \frac{\text{cm}^3}{\text{g s}} = \frac{\text{erg cm}}{\text{g}^2}$$

$$[\text{erg}] = \frac{\text{cm}^2}{\text{s}^2} \text{g}$$

Contrary to, for example, molecular forces, gravity is a long range force, i.e.: **we cannot neglect distant regions**

Illustration: an homogeneous medium ($\rho(\vec{r}) = \rho_0$)

Force on point O due to a thin shell of mass Δm at distance r

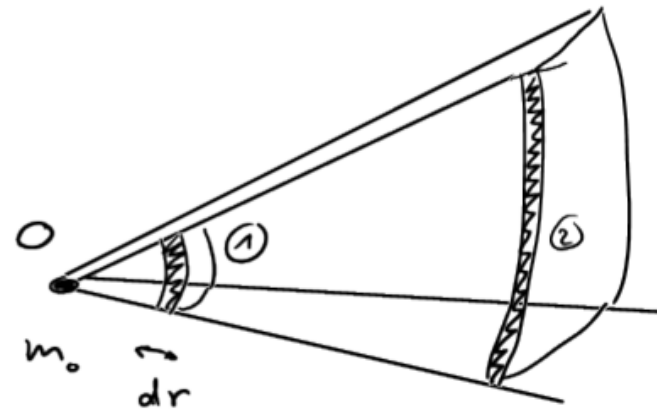
$$\Delta F = \frac{G m_0 \Delta m}{r^2}$$

but $\Delta m = \rho r^2 \Delta \Omega dr$

thus

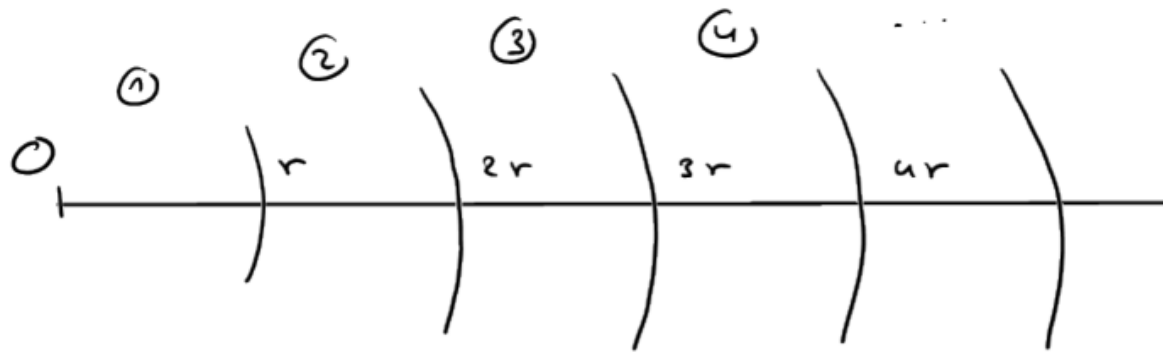
$$\Delta F = G m_0 \rho \Delta \Omega dr$$

etc indep. of r



solid angle $\Delta \Omega$

Split the space in shells of thickness r



$$\Delta F_1 = \int_0^r \Delta F = \int_0^r dr G m_0 \rho \Delta R = G m_0 \rho \Delta R r$$

$$\Delta F_2 = \int_r^{2r} \Delta F = \dots = G m_0 \rho \Delta R r$$

As the contribution of all shell is the same, the contribution of the stars with $r' > r$ will dominate over the ones with $r' < r$

We cannot neglect regions at large distances !

Corollary : As the force is dominated by the mass at large scale, the force varies smoothly along the trajectory of a particle (star).

Stellar systems can be modeled by smooth mass distributions



Notes

① 2D case $\delta F = \frac{Gm_0}{r^2} \delta m = \frac{Gm_0}{r^2} \Sigma \delta \theta r dr = Gm_0 \Sigma \delta \theta dr \frac{1}{r}$
 Σ : surface density

$$\delta F_0 = \int_0^r Gm_0 \rho \delta \theta \frac{1}{r} dr = Gm_0 \rho \delta \theta [\ln(r) - \ln(0)] = \infty$$

(initial) r

$$\delta F_i = \int_{ir} Gm_0 \rho \delta \theta \frac{1}{r} dr = Gm_0 \rho \delta \theta [\ln(i+1) - \ln(i)] < \infty$$

= 0 (i → ∞)

1D case $\delta F = \frac{Gm_0}{r^2} \delta m = \frac{Gm_0}{r^2} \lambda dr$ λ : linear density

$$\delta F_0 = \int_0^r Gm_0 \lambda \frac{1}{r^2} dr = Gm_0 \lambda \left[-\frac{1}{r} + \frac{1}{\infty} \right] = \infty$$

(initial) r

$$\delta F_i = \int_{ir} Gm_0 \lambda \frac{1}{r^2} dr = Gm_0 \lambda \left[\frac{1}{r} \left(\frac{1}{i} - \frac{1}{i+1} \right) \right] < \infty$$

= 0 (i → ∞)

② Molecular dynamics

"long distance" attraction force
between two molecules,

Van der Waals Force $\sim r^{-7}$

\Rightarrow local molecules dominates

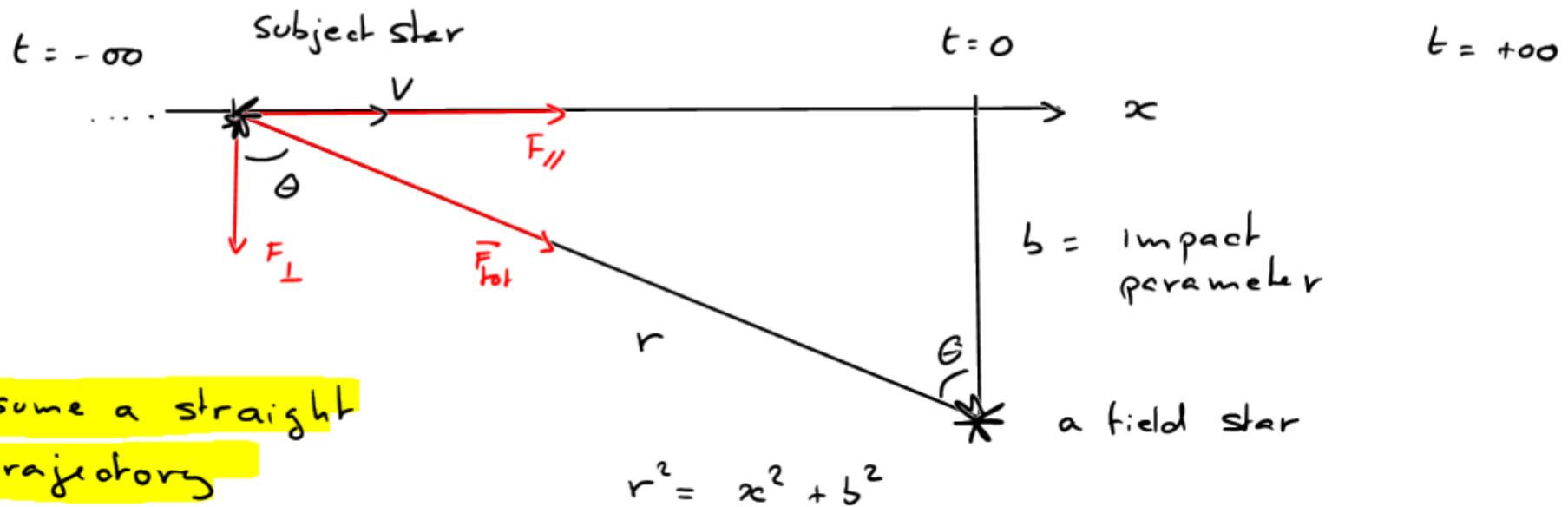
Relaxation Time

Question :

How accurate is the assumption that a galaxy may be modelled as a smooth distribution ?

- 1) Effect of one star on the orbit of a peculiar star.
- 2) Effect of all stars of a stellar system on a peculiar star.
- 3) Under which conditions the orbit of a peculiar star is strongly influenced by the discrete nature of the stellar system (importance of “collision” with other stars).

- ① Estimate the effect of one star on the trajectory of a peculiar star
-



We assume a straight line trajectory

- 1) acceleration along x (F_{\parallel}) does not matter, as it is symmetric (the star decelerate after passing the field star)
- 2) acceleration perpendicular to x (F_{\perp})

$$\begin{aligned}
 |F_{\perp}| &= |F_{\text{tot}}| \cos \Theta = \frac{Gmm}{r^2} \cos \Theta && \text{but } \cos \Theta = \frac{b}{r} \\
 &= \frac{Gmm}{x^2 + b^2} \frac{b}{\sqrt{x^2 + b^2}} = \frac{Gmm}{(x^2 + b^2)^{3/2}} b \\
 & && r^2 = x^2 + b^2 \\
 &= \frac{Gmm}{\left(1 + \frac{x^2}{b^2}\right)^{3/2}} \frac{1}{b^2} = \frac{Gmm}{b^2} \left(1 + \frac{x^2}{b^2}\right)^{-3/2}
 \end{aligned}$$

with $x = vt$

$$F_{\perp} = |F_{\perp}| = \frac{Gmm}{b^2} \left(1 + \left(\frac{vt}{b}\right)^2\right)^{-3/2}$$

Newton 2nd law

$$\begin{aligned}
 F_{\perp} &= m a_{\perp} = m \frac{dV_{\perp}}{dt} \\
 dV_{\perp} &= \frac{F_{\perp}}{m} dt
 \end{aligned}$$

Integrating over time from $t = -\infty$ to $t = \infty$

$$\Delta V_{\perp} = \frac{1}{m} \int_{-\infty}^{\infty} F_{\perp} dt \quad \text{net velocity increase}$$

$$= \frac{1}{m} \int_{-\infty}^{\infty} \frac{GmM}{b^2} \left(1 + \left(\frac{vt}{b}\right)^2\right)^{-\frac{3}{2}} dt$$

with : $s = \frac{vt}{b}$ $ds = \frac{v}{b} dt$

$$\Delta V_{\perp} = \frac{Gm}{b^2} \int_{-\infty}^{\infty} \left(1 + s^2\right)^{-\frac{3}{2}} dt = 2 \frac{Gm}{bv}$$

This can be written as :

$$\delta V_{\perp} = \frac{Gm}{b^2} \cdot \frac{2b}{v}$$

$$= 2 \frac{Gm}{bv}$$

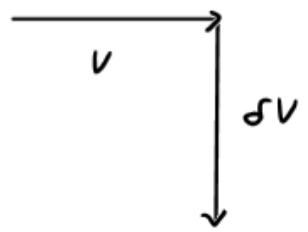
acceleration
at the closest
approach

"duration"
of the closest
approach

Note : our hypothesis of a straight line is ok if $\frac{\delta V}{V} \ll 1$

$$\Rightarrow b \gg \frac{2Gm}{v^2} := b_{90}$$

b_{90} define
as $V = \delta V$



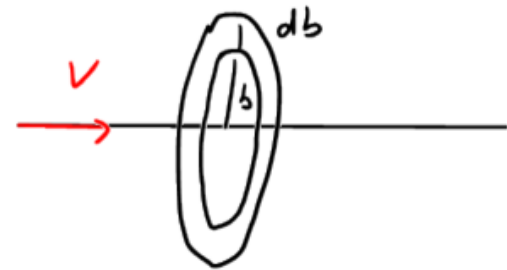
② Effect of all stars of a stellar system on the trajectory of a peculiar star

N : total number of stars

R typical size of the system

- number density of stars per unit of surface $n = \frac{N}{\pi R^2}$
- number of stars met by the star with $[b, b + db]$

$$\begin{aligned}\delta N &= 2\pi b db \cdot n \\ &= 2\pi b db \frac{N}{\pi R^2} = \frac{2N b db}{R^2}\end{aligned}$$

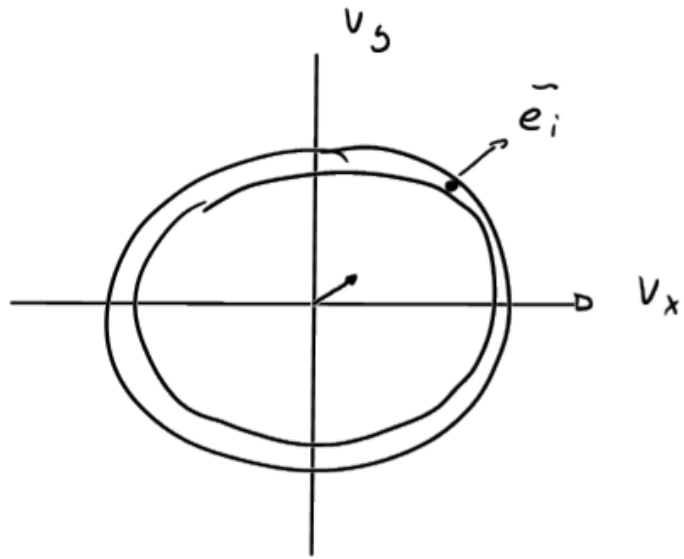


Note: each of those stars (alone) induces a change of velocity

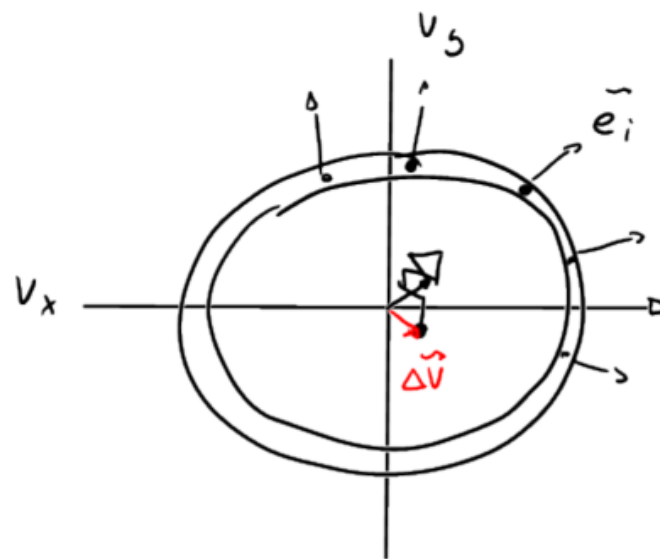
$$\delta V_{\perp} = \frac{2 G m}{b v}$$

Velocity change due to δN stars in a ring

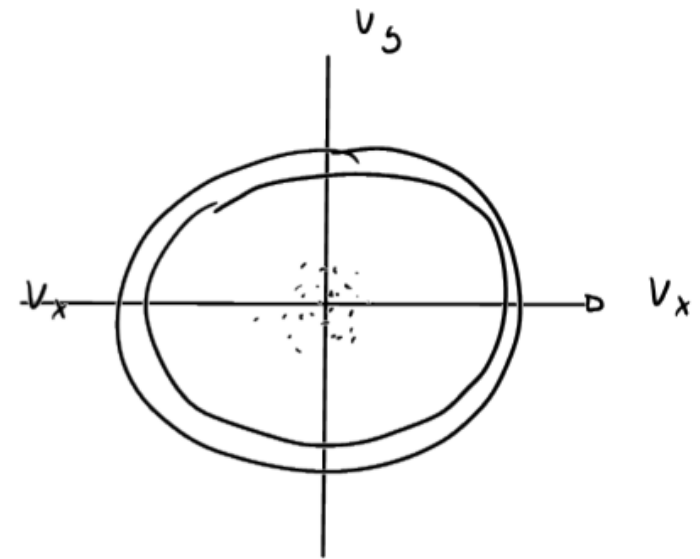
$$\Delta \vec{v} = \sum_i \delta v_{\perp,i} \vec{e}_i$$



velocity change due to one star in the ring



velocity change due to δN stars in the ring



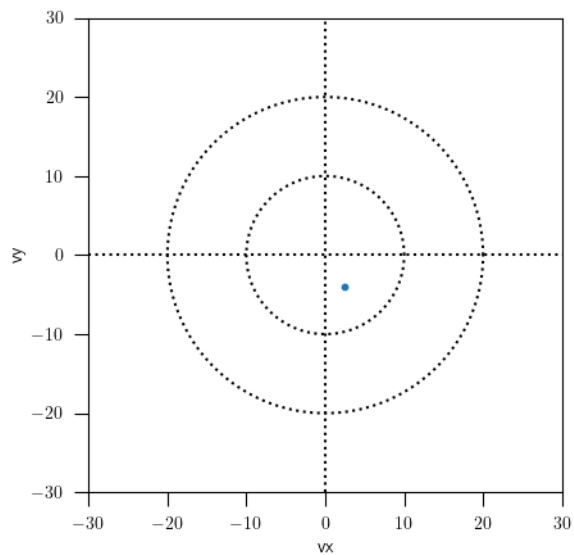
Statistical distribution of velocity changes

$$\Delta \vec{v} = \sum_i \delta v_{\perp,i} \vec{e}_i = 0$$

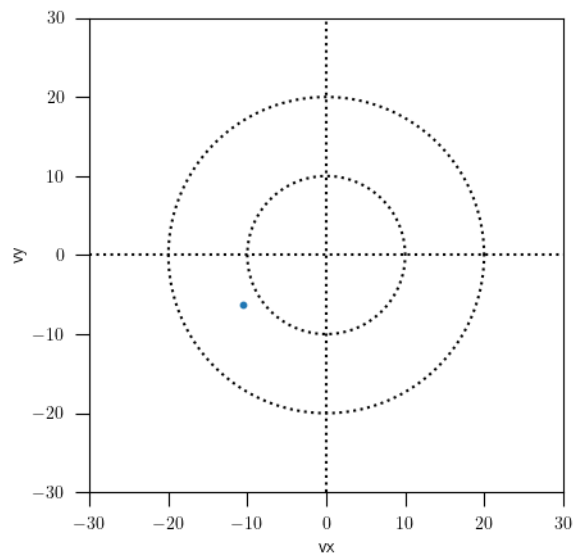
$$\begin{aligned} \Delta v^2 &= \sum_i \delta v_{\perp,i}^2 \neq 0 \\ &= \delta N \delta v^2 = \frac{2N b db}{R^2} \left(\frac{2Gm}{bv} \right)^2 = \frac{8NG^2 m^2}{v^2 R^2} \frac{db}{b} \end{aligned}$$

Spread in the velocity space due to two-body encounters

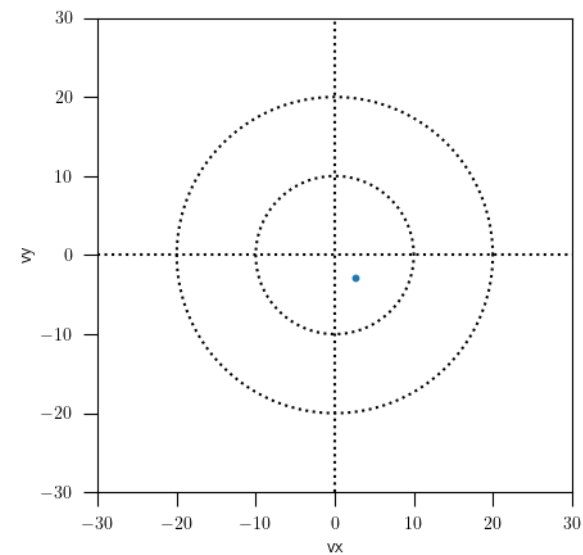
N=1



N=1

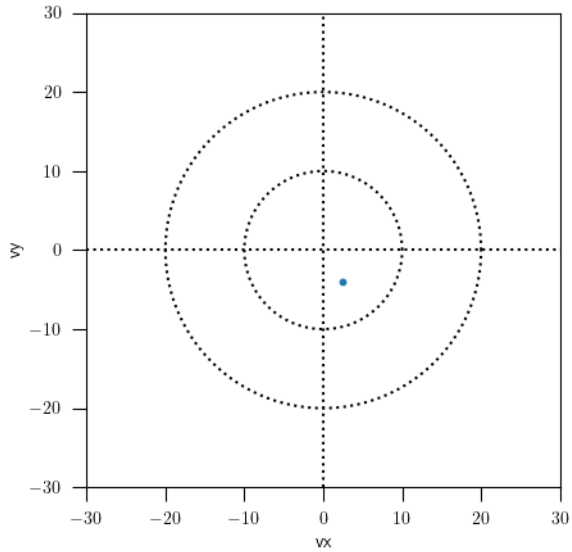


N=1

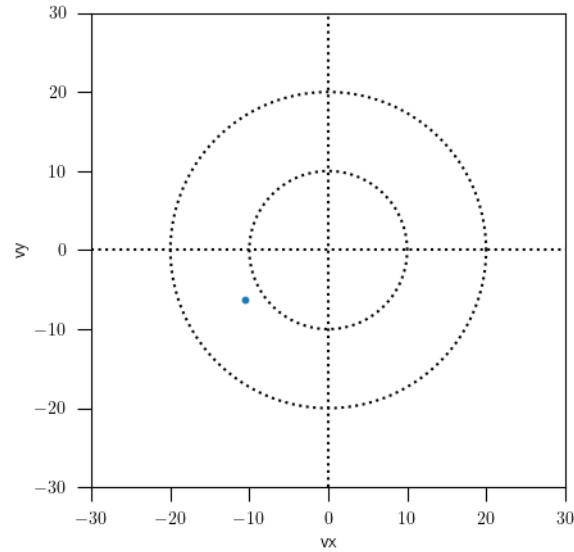


Spread in the velocity space due to two-body encounters

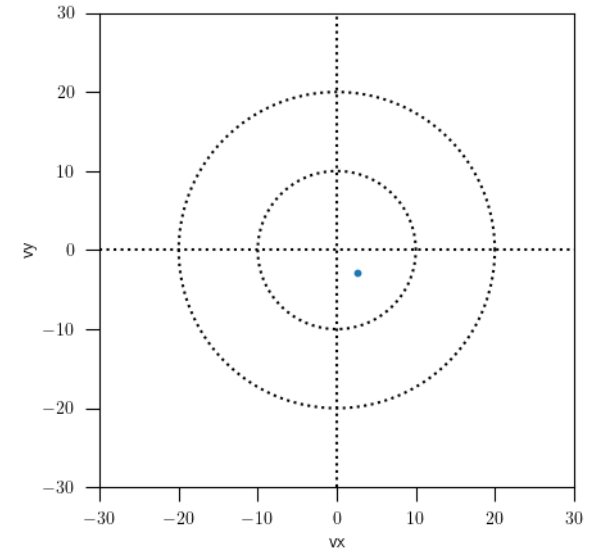
N=1



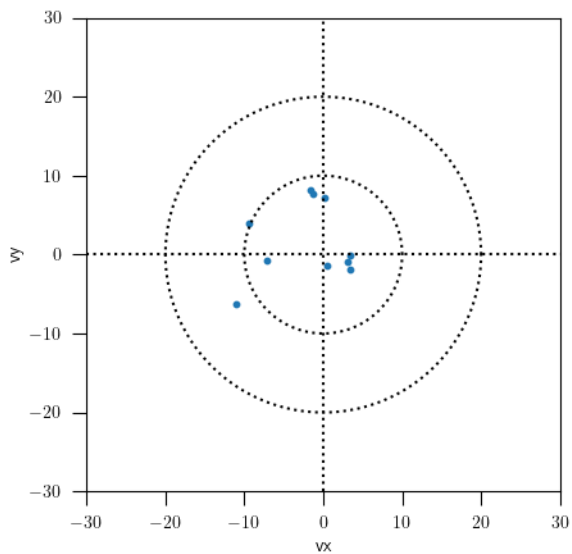
N=1



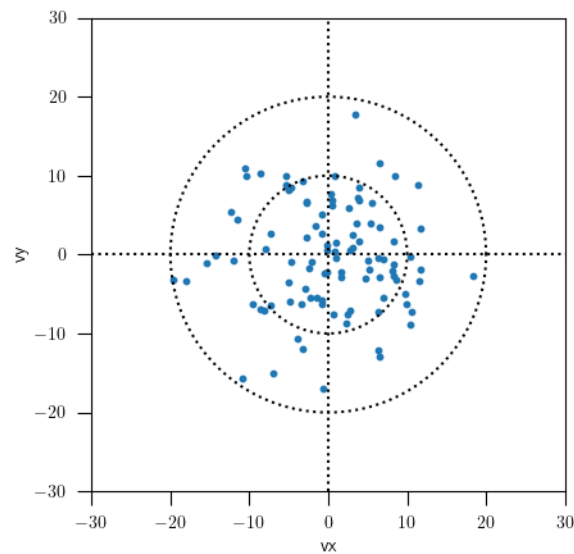
N=1



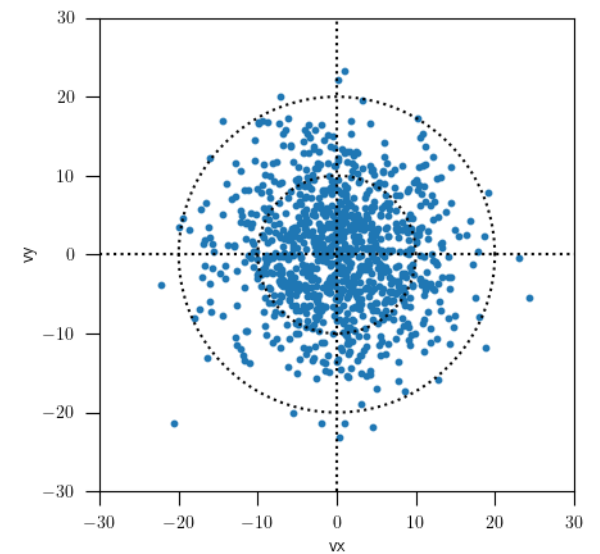
N=10



N=100



N=1000



For all encounters, we integrate over b from b_{\min} to b_{\max}

$$b_{\min} := \beta_1 b_{s0} \quad \text{if } b < b_{s0} \quad \Delta V \sim v \quad \beta_1 \approx 1$$

$$b_{\max} := \beta_2 R \quad \text{if } b > R \quad \text{the density is no longer constant} \quad \beta_2 \approx 1$$

we get

ln : Coulomb logarithm

$$\Delta V^2 = 8N \left(\frac{Gm}{vR} \right)^2 \int_{b_{\min}}^{b_{\max}} \frac{db}{b} = 8N \left(\frac{Gm}{vR} \right)^2 \ln \left(\frac{b_{\max}}{b_{\min}} \right)$$

$$\Delta V^2 = 8N \left(\frac{Gm}{vR} \right)^2 \left[\ln \left(\frac{R}{b_{s0}} \right) + \ln \left(\frac{\beta_2}{\beta_1} \right) \right]$$

- variation due to one crossing.

≈ 0

Replacing R with v

Typical velocity of one star (circular orbit)

$$v^2 \sim \frac{GNm}{R} \Rightarrow R = \frac{GNm}{v^2}$$

$$\Delta v^2 = 8N \left(\frac{Gm}{vR} \right)^2 \ln \Lambda = 8N \left(\frac{v}{N} \right)^2 \ln \Lambda = 8 \frac{v^2}{N} \ln \Lambda$$

$$\Delta v^2 = 8 \frac{v^2}{N} \ln \Lambda$$

Crossing time

Time for the star to cross the system

$$t_{\text{cross}} \approx \frac{R}{v}$$

$$\left(v^2 \sim \frac{GM_{\text{in}}}{R} \right)$$

- Each time a star will cross the system, the square of its velocity will change by an amount Δv^2

$$n_{\text{cross}} \Rightarrow$$

$$\Delta v_{n_{\text{cross}}}^2 = n_{\text{cross}} \cdot v^2 \frac{8}{N} \ln \Lambda$$

Relaxation

- n_{relax} : number of crossing time to significantly change the velocity, i.e. to have $\Delta v^2 \cong v^2$

$$n_{\text{relax}} \cdot v^2 \frac{8}{N} \ln \Lambda = v^2$$

$$n_{\text{relax}} = \frac{N}{8 \ln \Lambda}$$

Relaxation time

Time for n_{relax} crossing.

$$t_{\text{rdax}} = n_{\text{relax}} \cdot t_{\text{cross}} = \frac{N}{8 \ln 2} \cdot t_{\text{cross}}$$

≡ Time after which a star change significantly its orbit with respect to a smooth density field

If $t < t_{\text{rdax}}$: the perturbations of nearby stars does not matter \Rightarrow
collisionless system

Estimations for stellar systems

N, m, R

using

$$b_{50} = \frac{2Gm}{v^2} \quad \text{circular orbit: } a = \frac{v^2}{r} = \frac{GNm}{r^2} \Rightarrow v^2 = \frac{GNm}{r}$$

we get

$$b_{50} = \frac{2R}{N}$$

$$\frac{R}{b_{50}} = \frac{N}{2}$$

$$\ln \mathcal{L} \approx \ln N$$

$$t_{\text{rdax}} = \frac{N}{8 \ln \mathcal{L}} \cdot t_{\text{cross}} \Rightarrow$$

$$t_{\text{rdax}} \approx \frac{0.1 N}{\ln N} t_{\text{cross}}$$

$$\begin{aligned} \ln \mathcal{L} &\approx \ln N \\ \frac{1}{8} &\approx 0.1 \end{aligned}$$

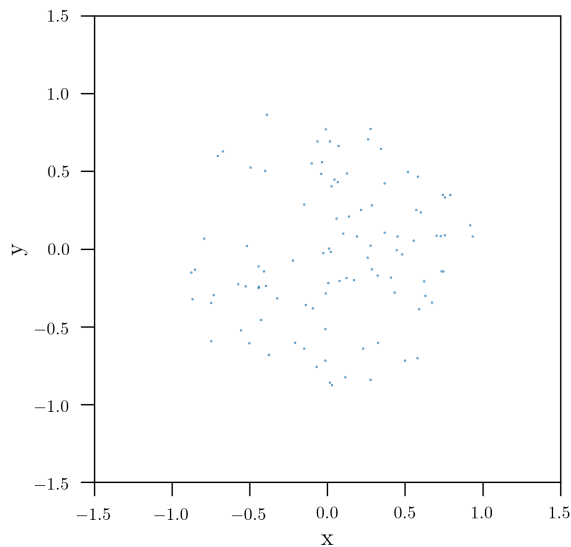
Numerical application / Relaxation Time

	N	R	b_{90}	$\ln(R/b_{90})$	t_{relax}
Globular Cluster	10^5	10 pc	2×10^{-4} pc	~ 10	~ 1 Gyr
Dwarf Galaxy	10^6	1 kpc	2×10^{-3} pc	~ 13	$\gg t_{\text{Hubble}}$
Spiral Galaxy	10^{10}	15 kpc	3×10^{-6} pc	~ 20	$\gg t_{\text{Hubble}}$
Galaxy Cluster	10^{13}	1 Mpc	2×10^{-7} pc	~ 30	$\gg t_{\text{Hubble}}$

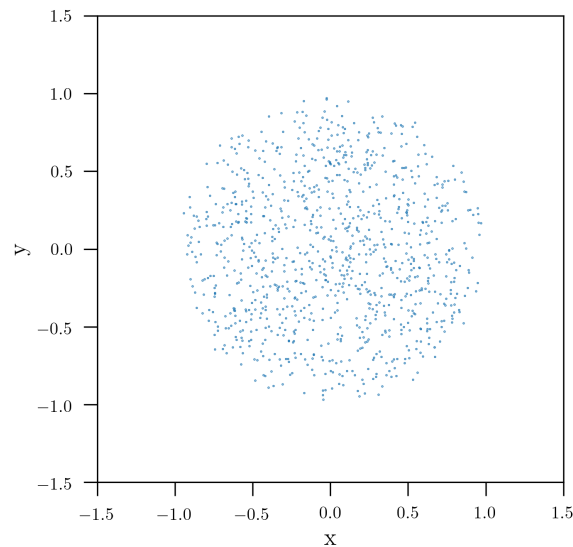
Numerical illustration

Orbit of a point mass in an homogeneous sphere
sampled with a discrete number of stars.

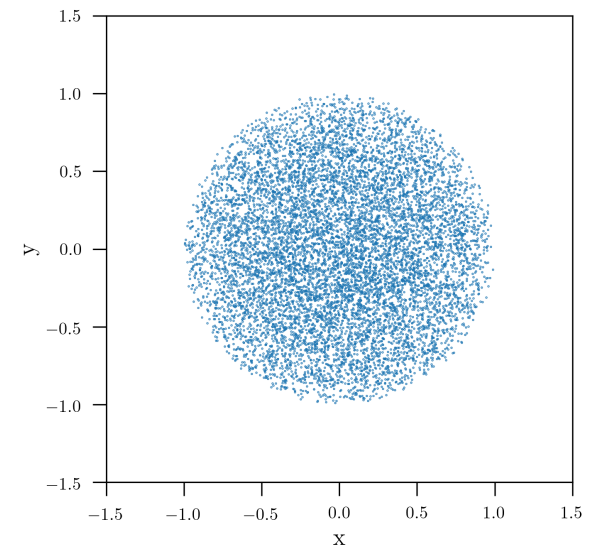
$N = 100$



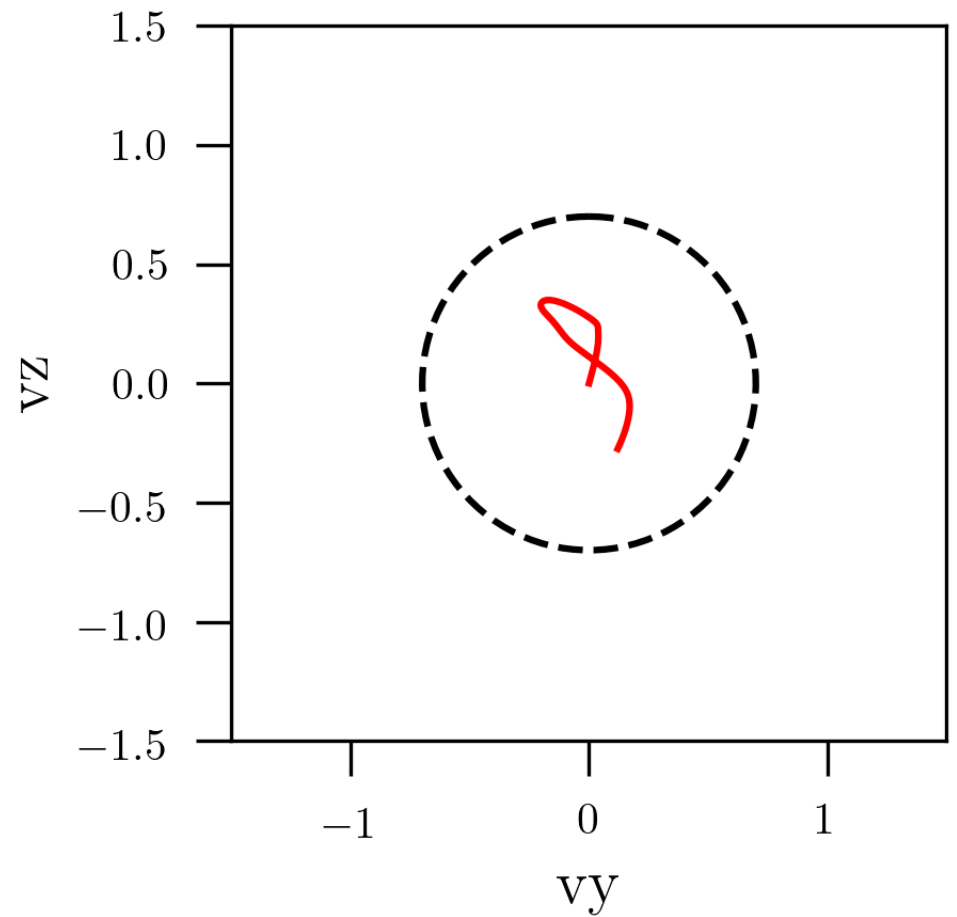
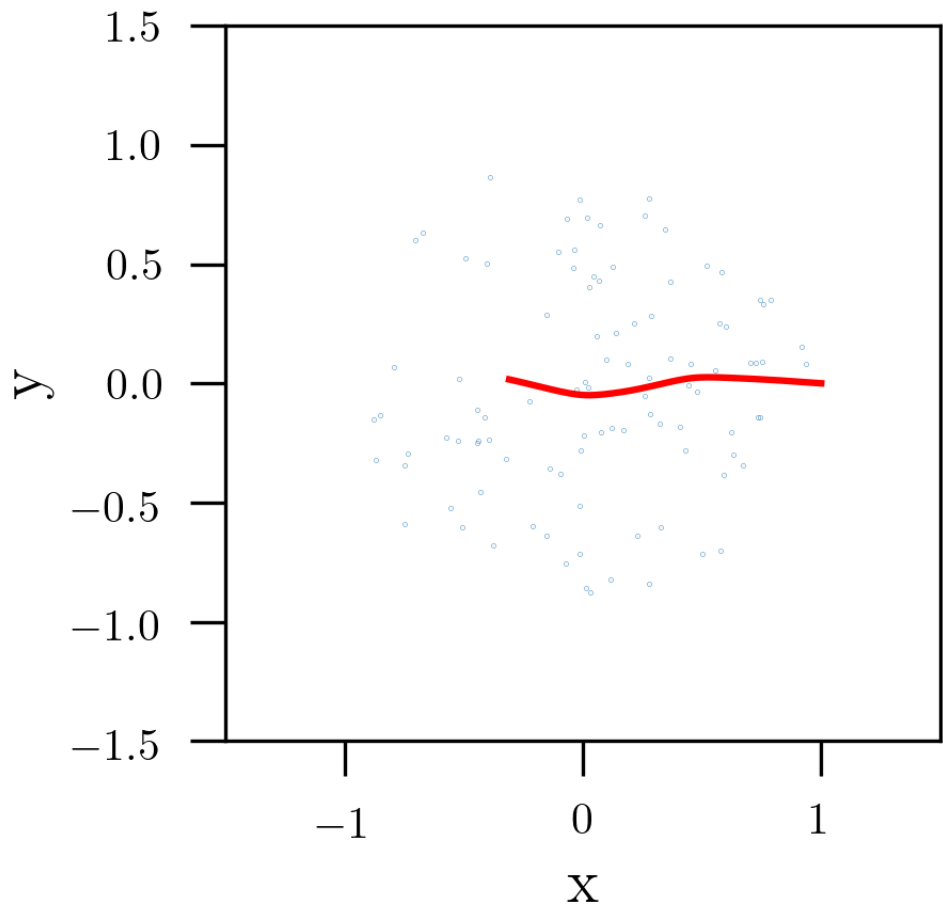
$N = 1000$



$N = 10000$

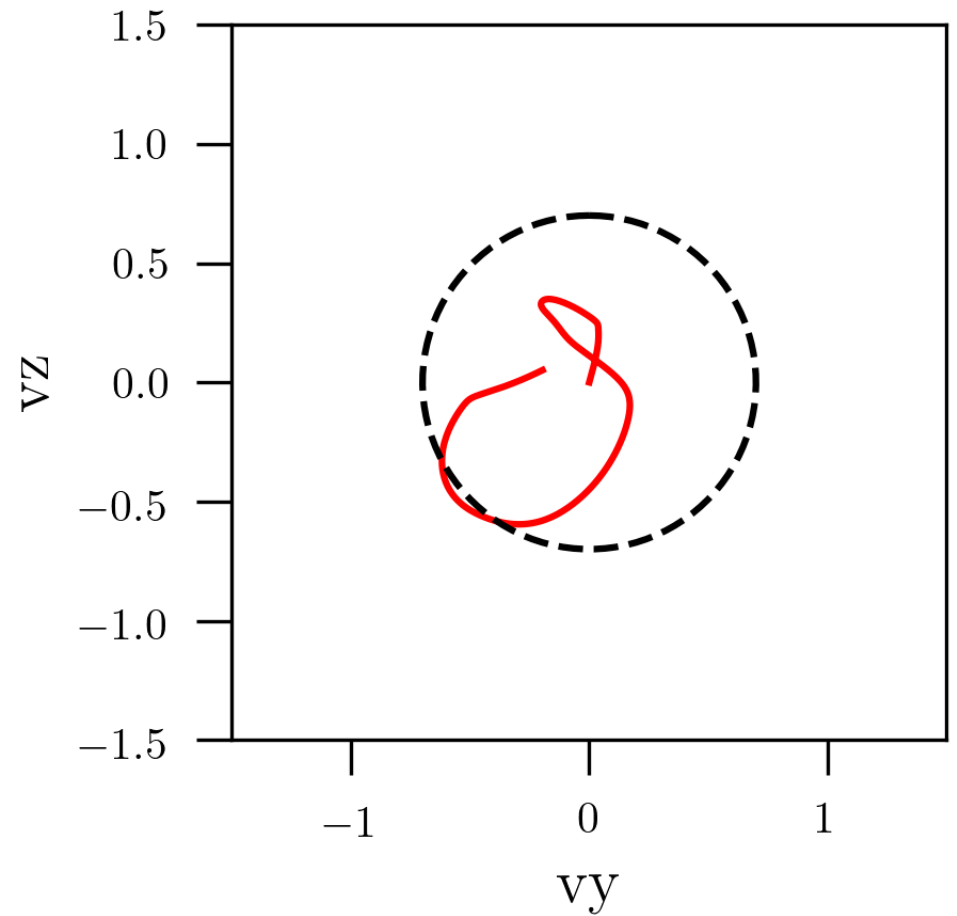
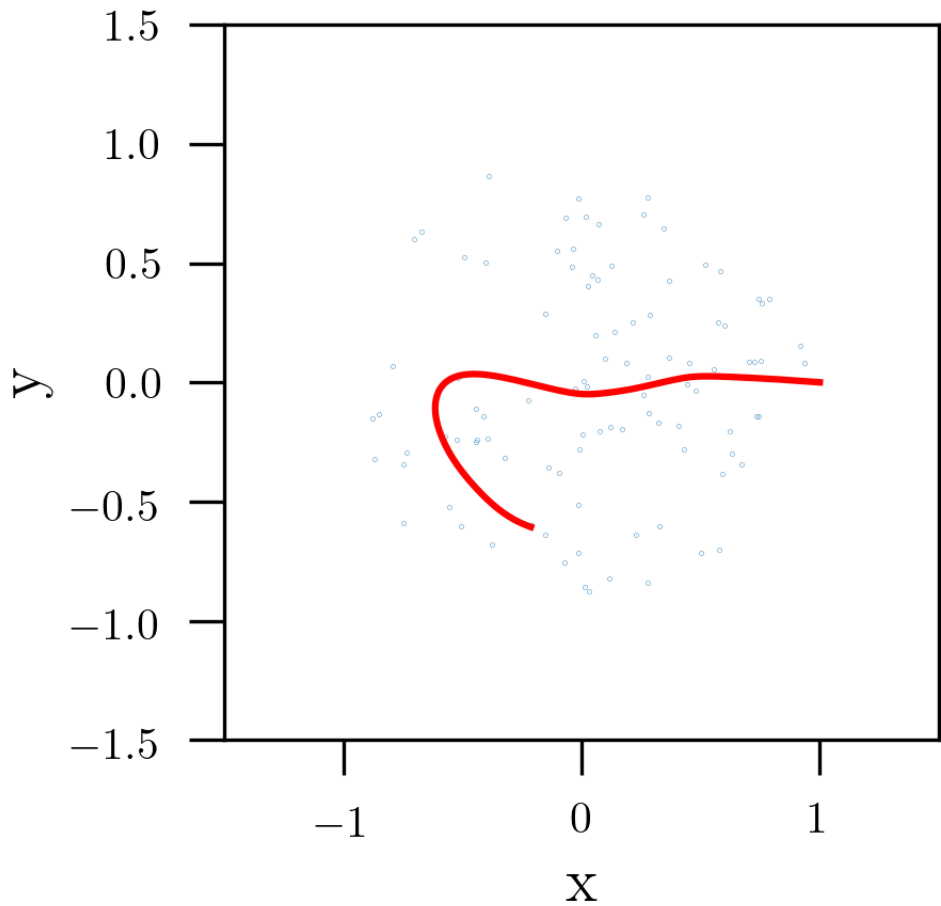


$N = 100$ Time = 2.00



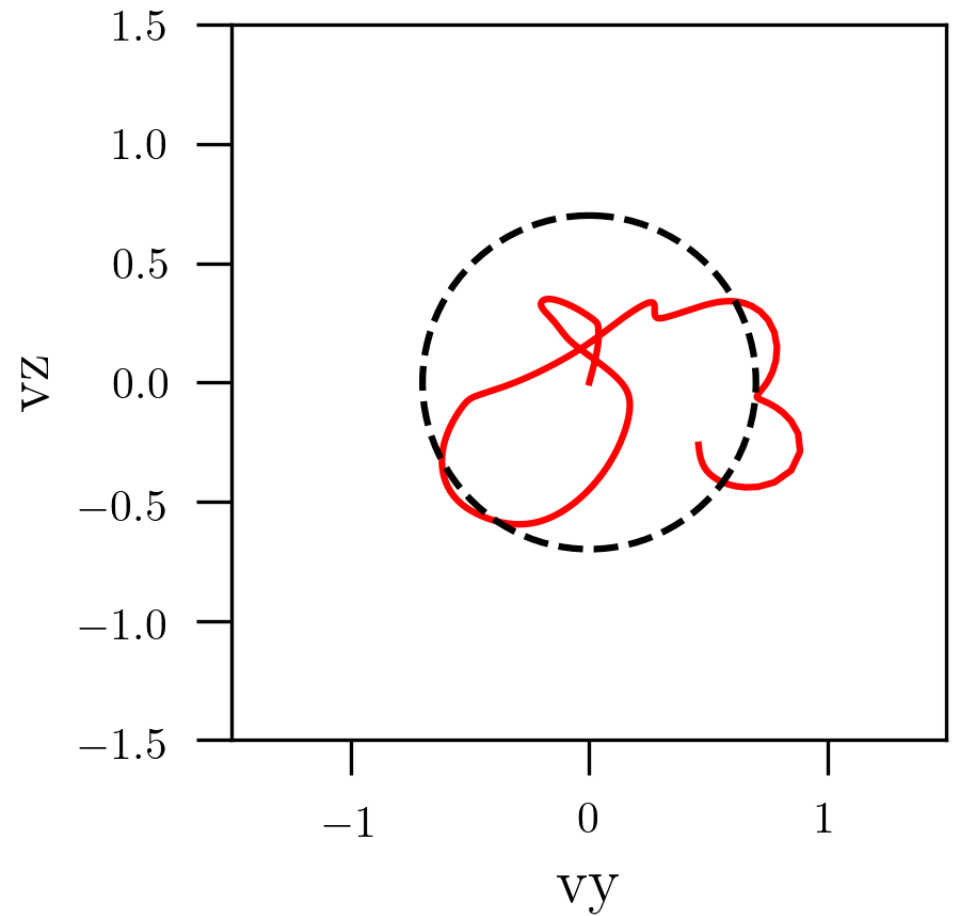
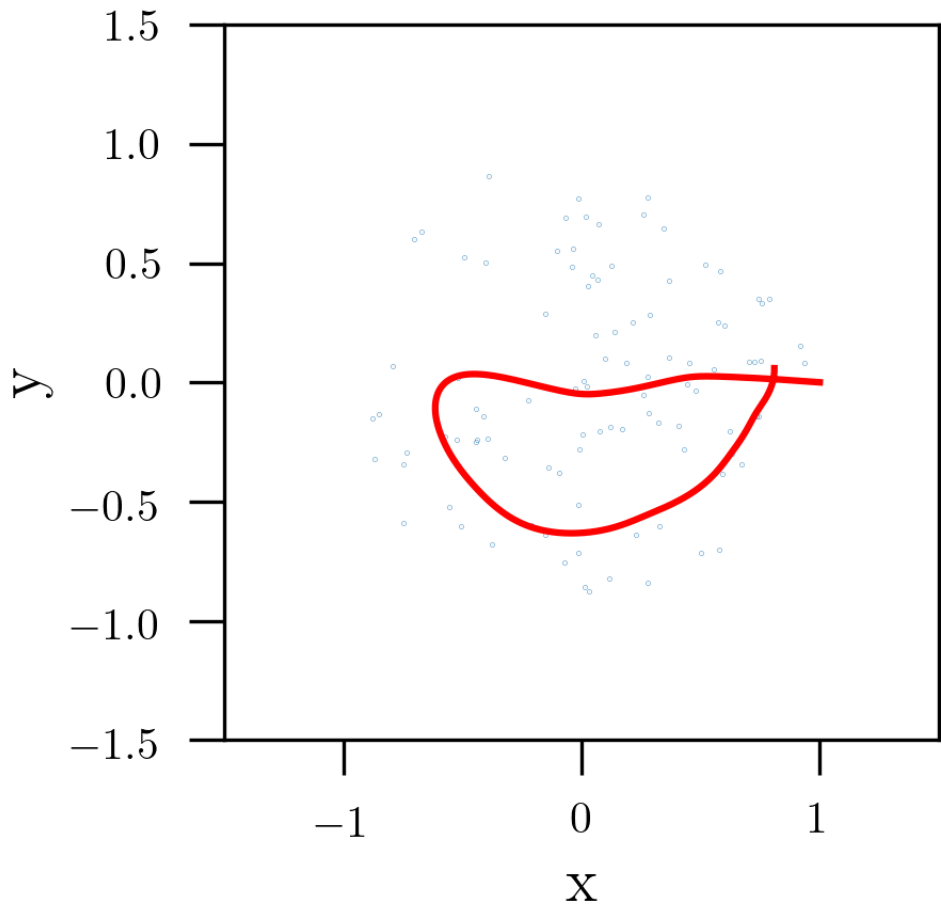
$t_{\text{relax}} = 3$

$N = 100$ Time = 4.00



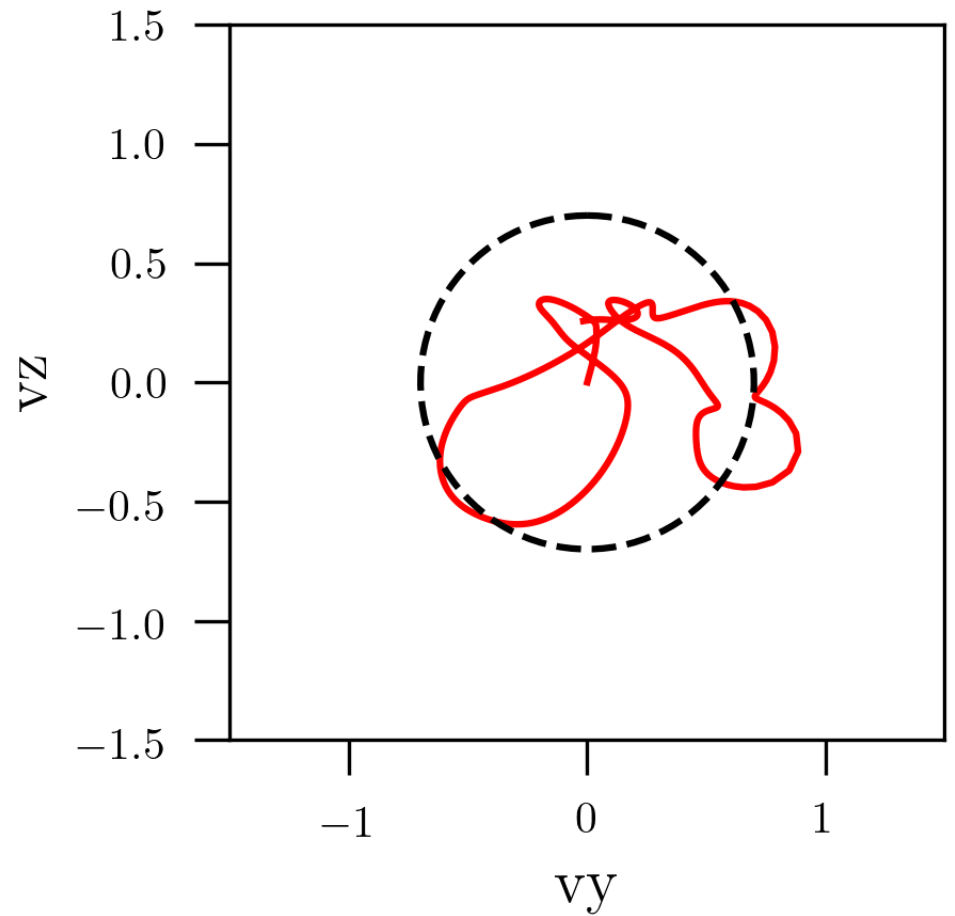
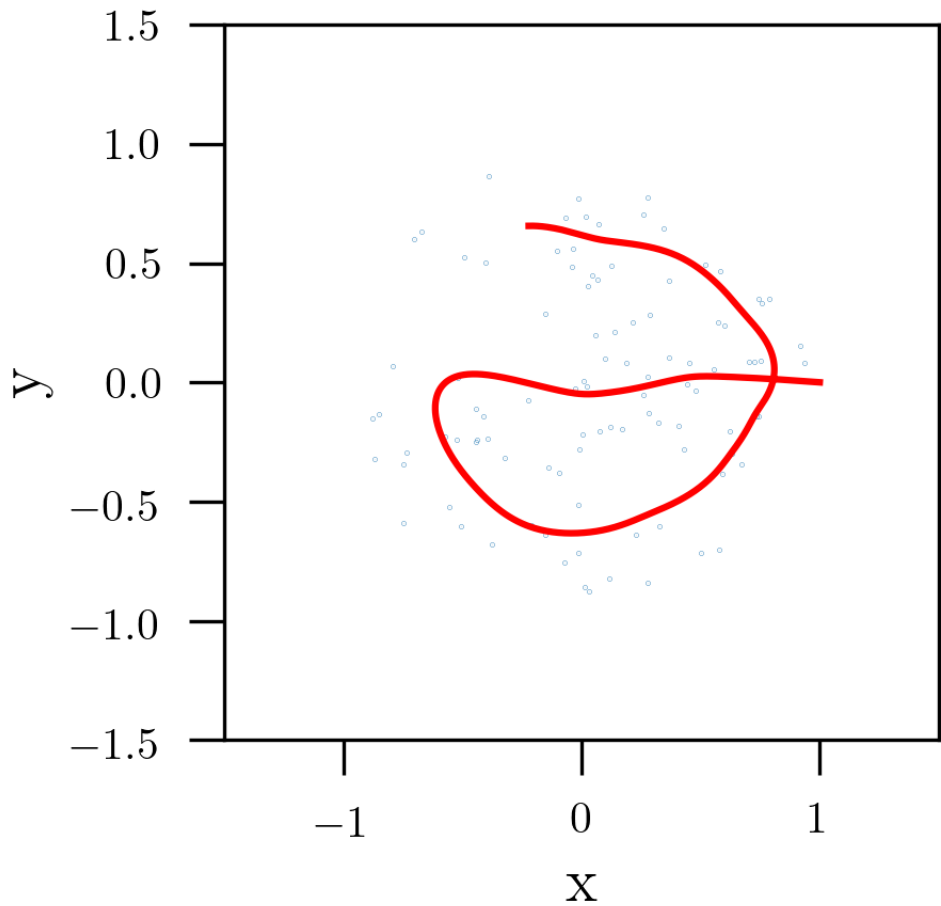
$t_{\text{relax}} = 3$

$N = 100$ Time = 6.00



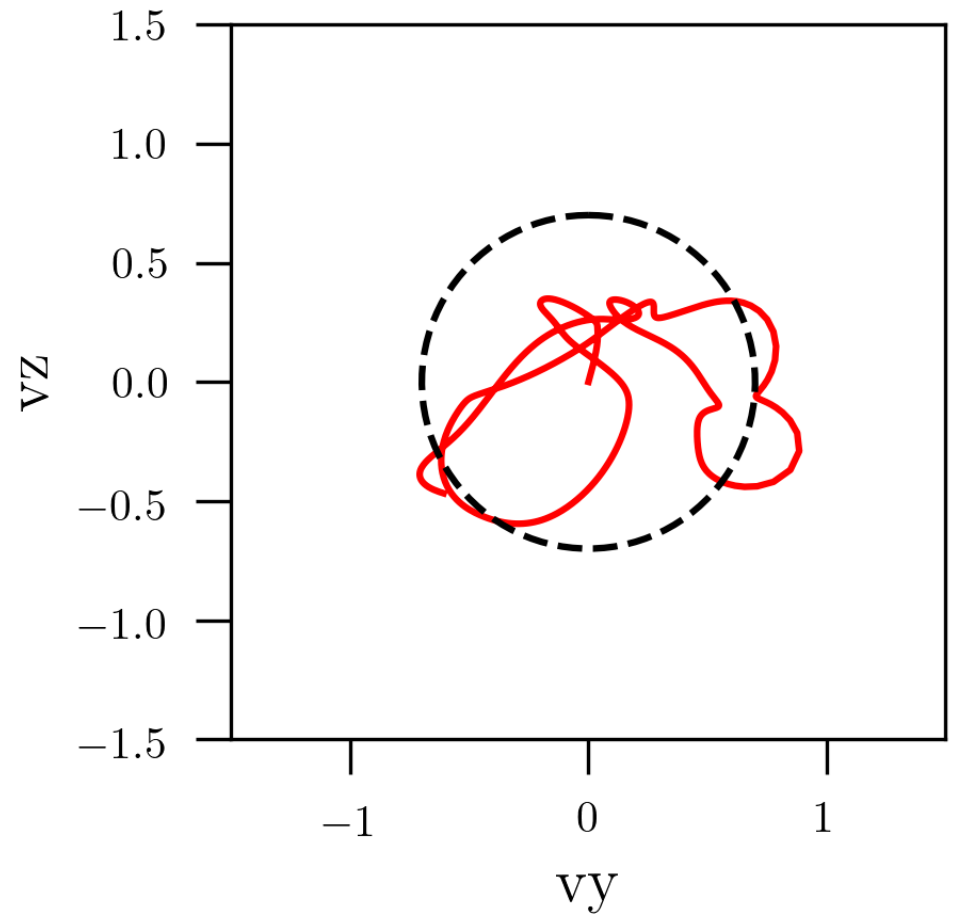
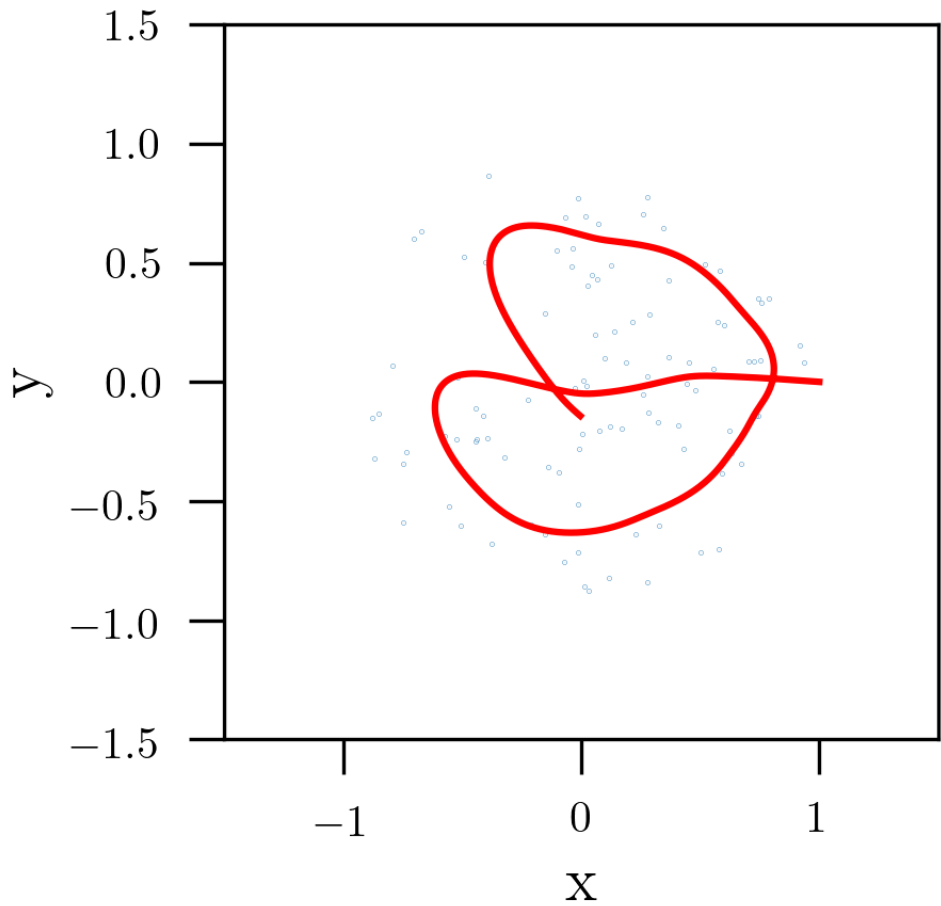
$t_{\text{relax}} = 3$

$N = 100$ Time = 8.00



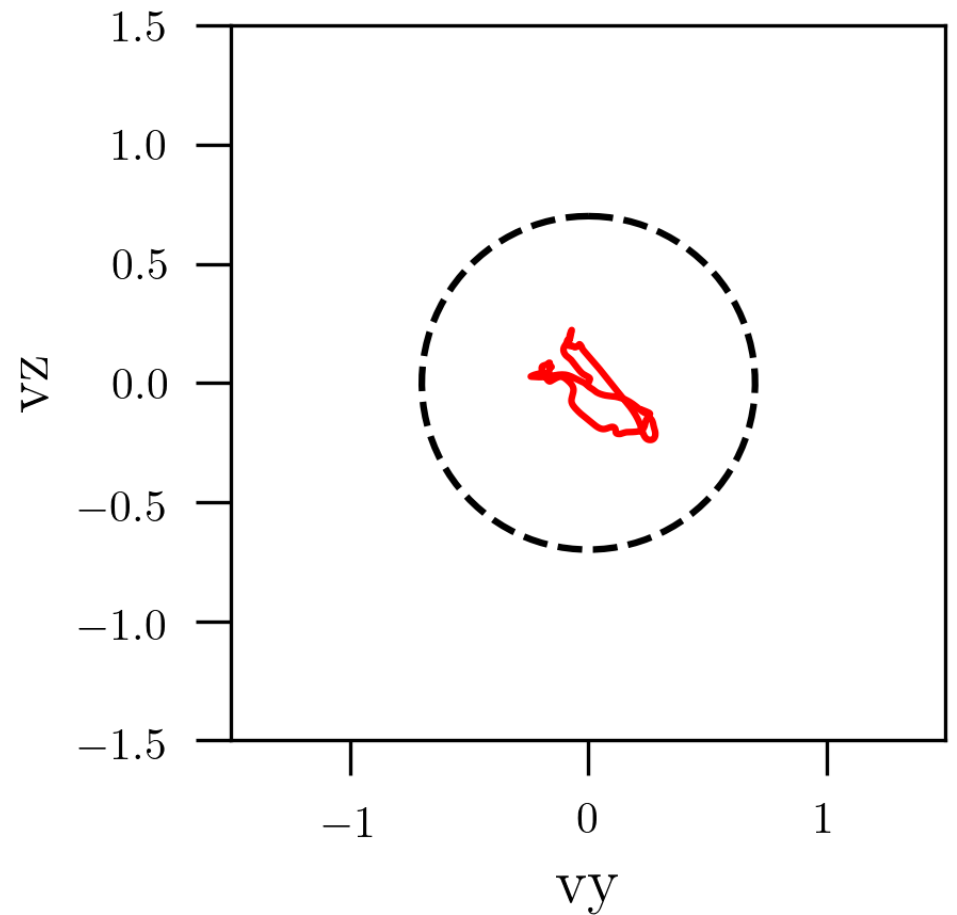
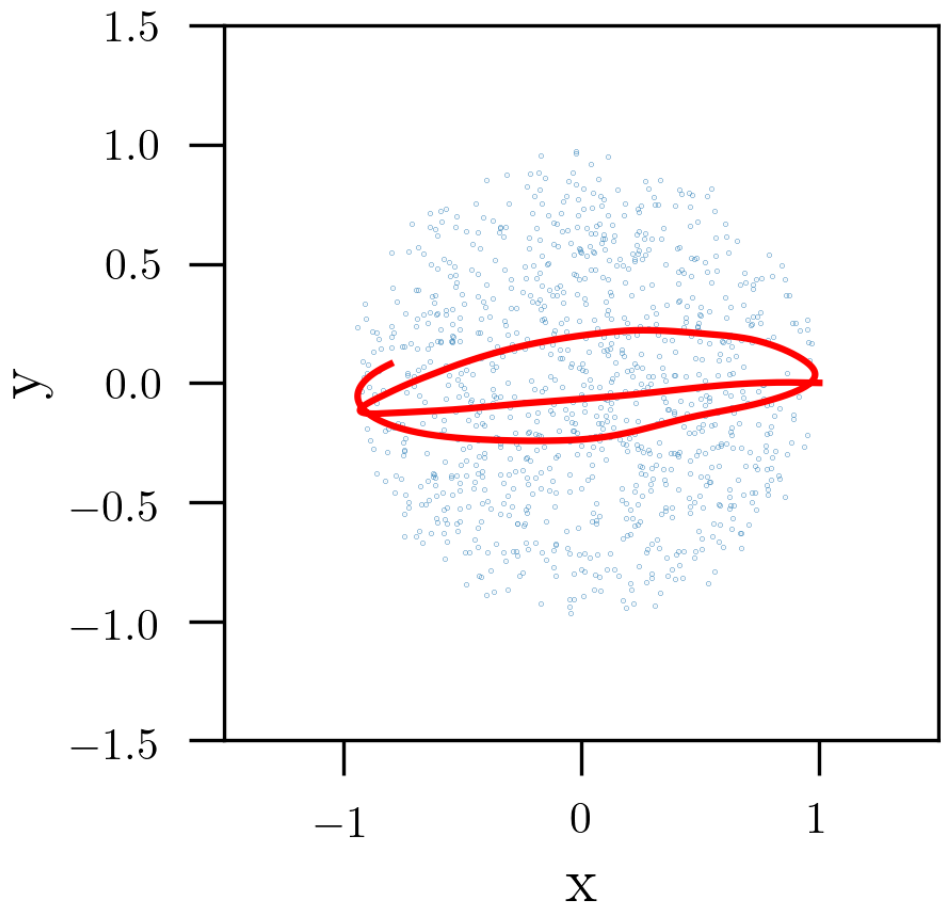
$t_{\text{relax}} = 3$

$N = 100$ Time = 10.00



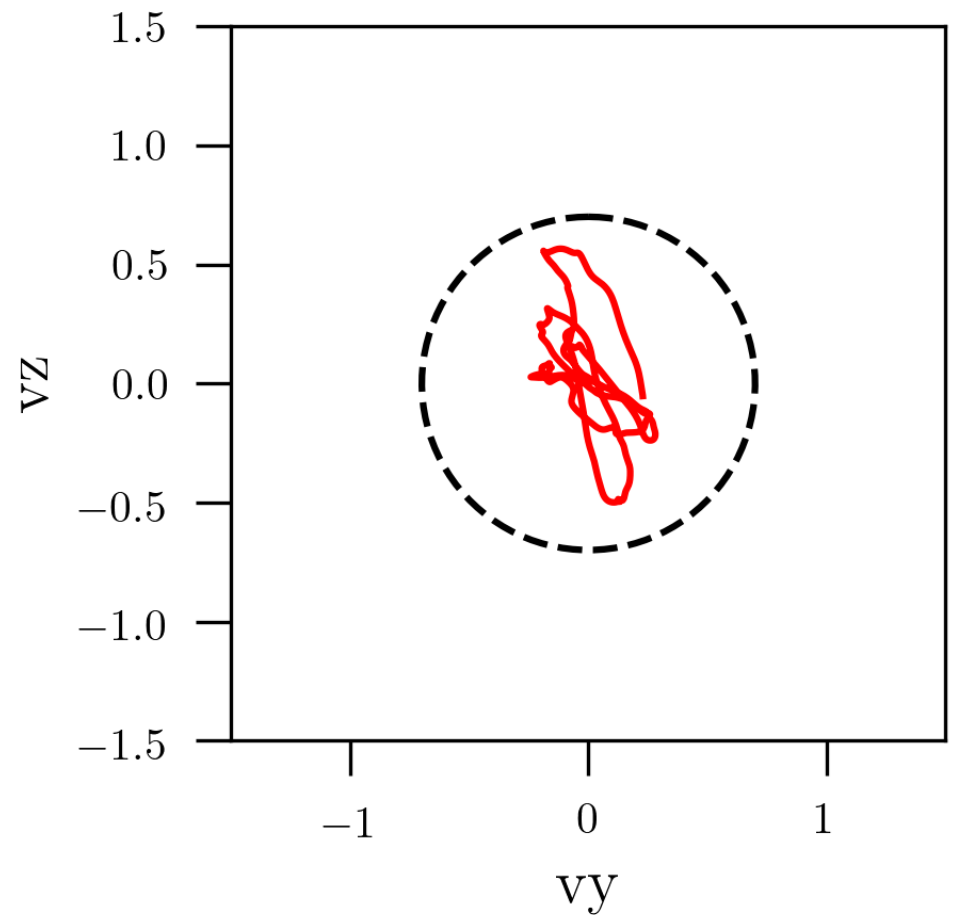
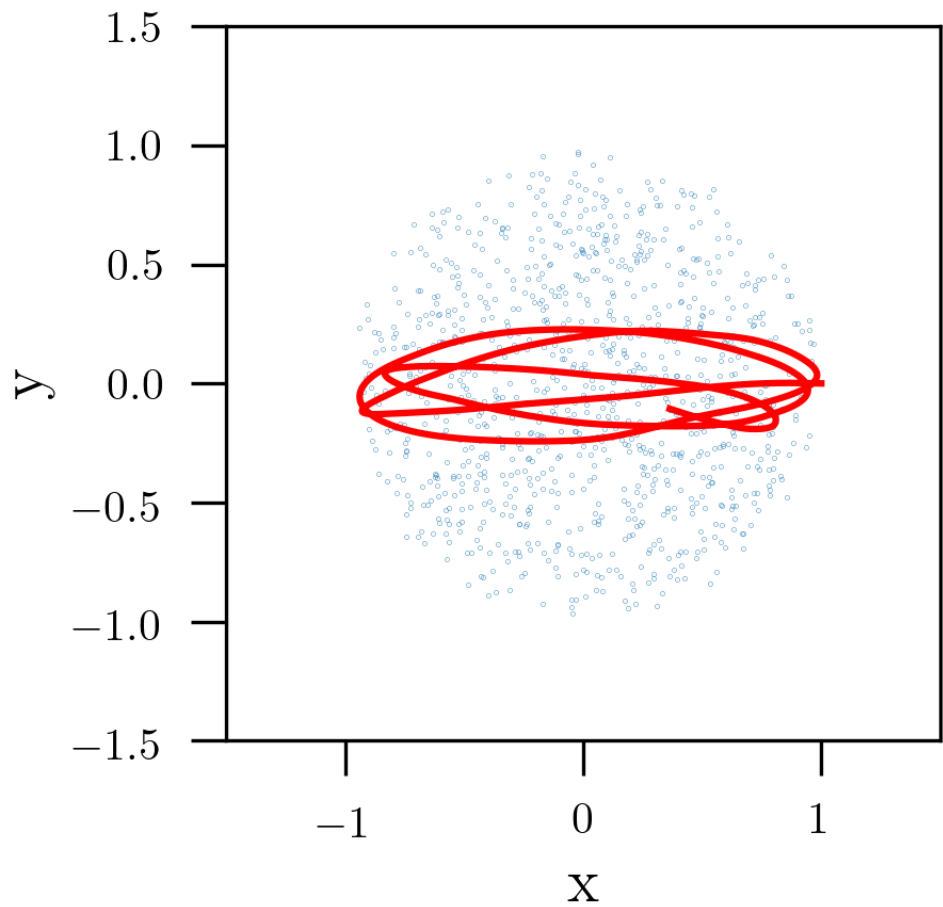
$t_{\text{relax}} = 3$

$N = 1000$ Time = 10.00



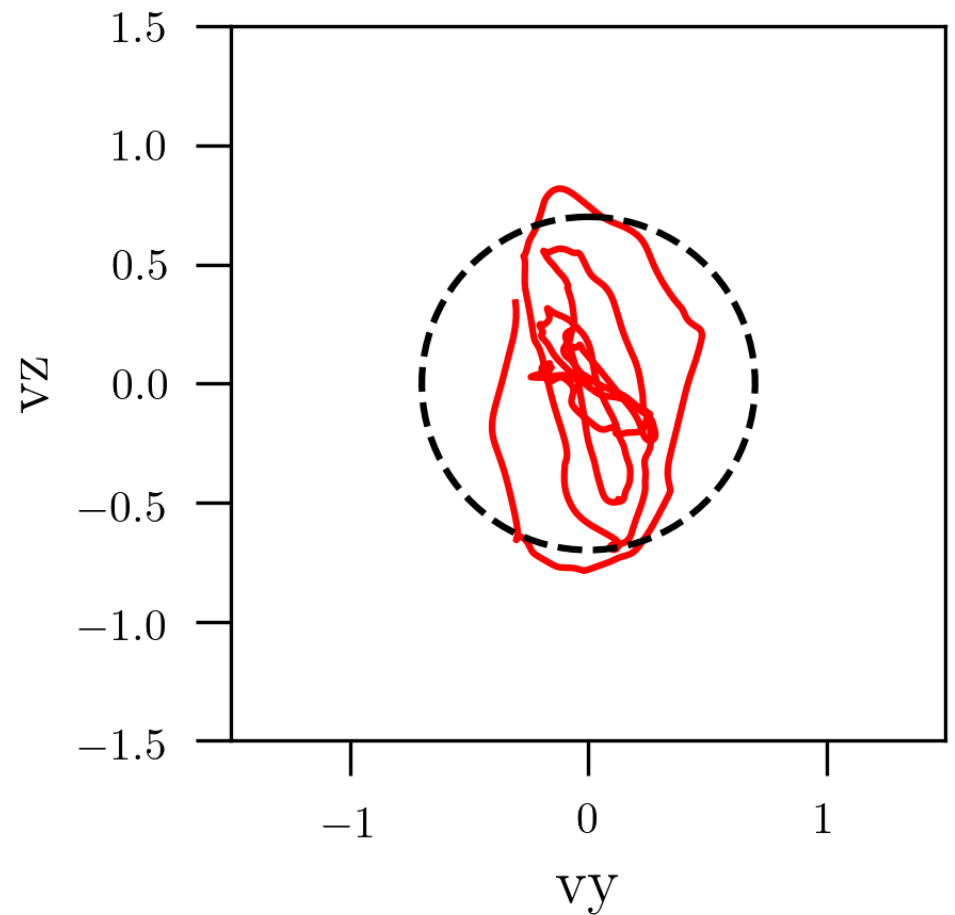
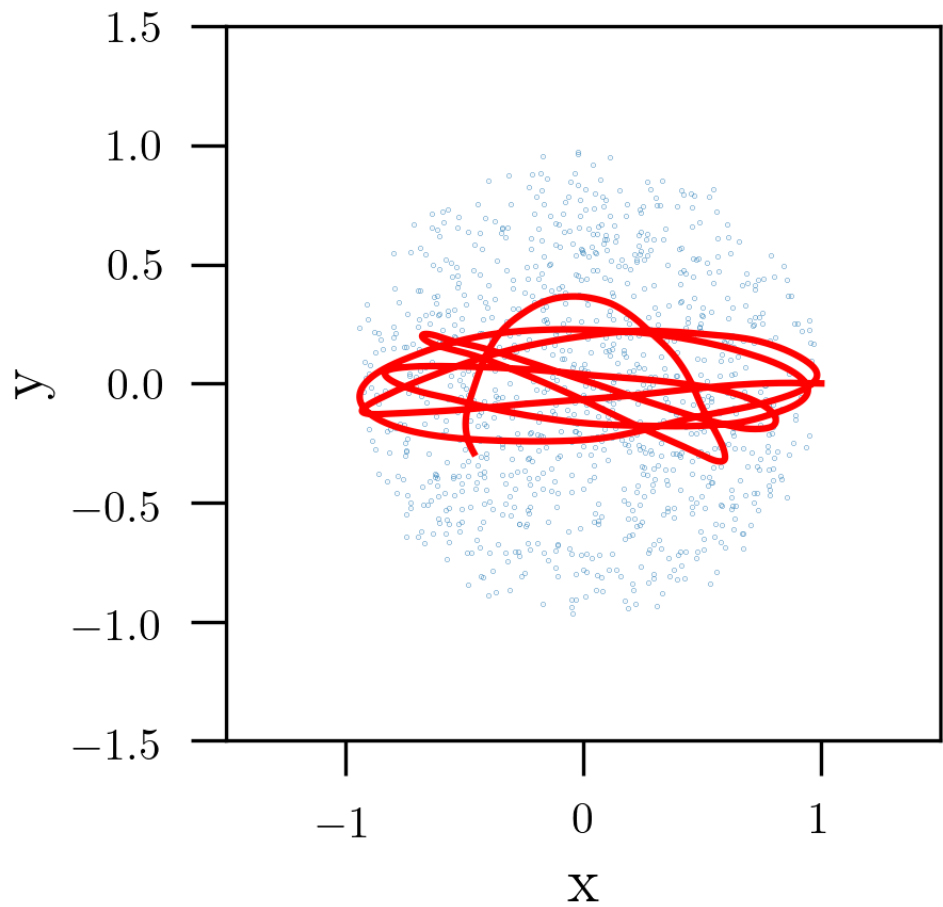
$t_{\text{relax}} = 20$

$N = 1000$ Time = 20.00



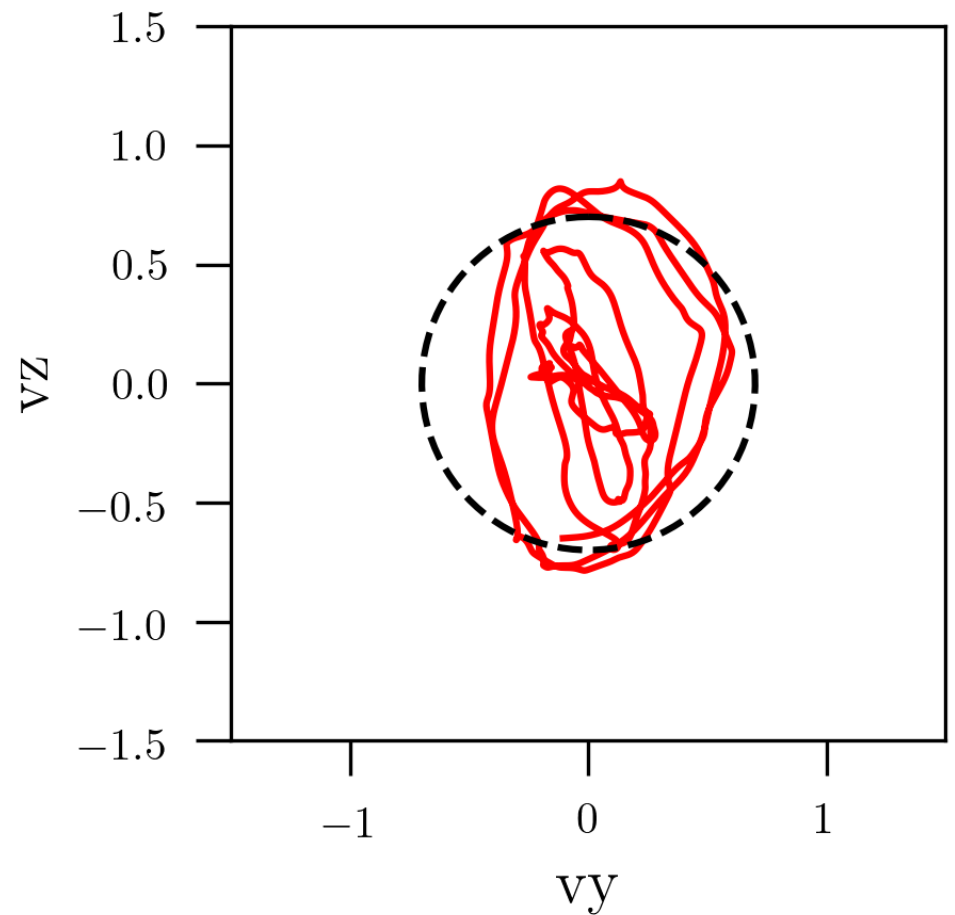
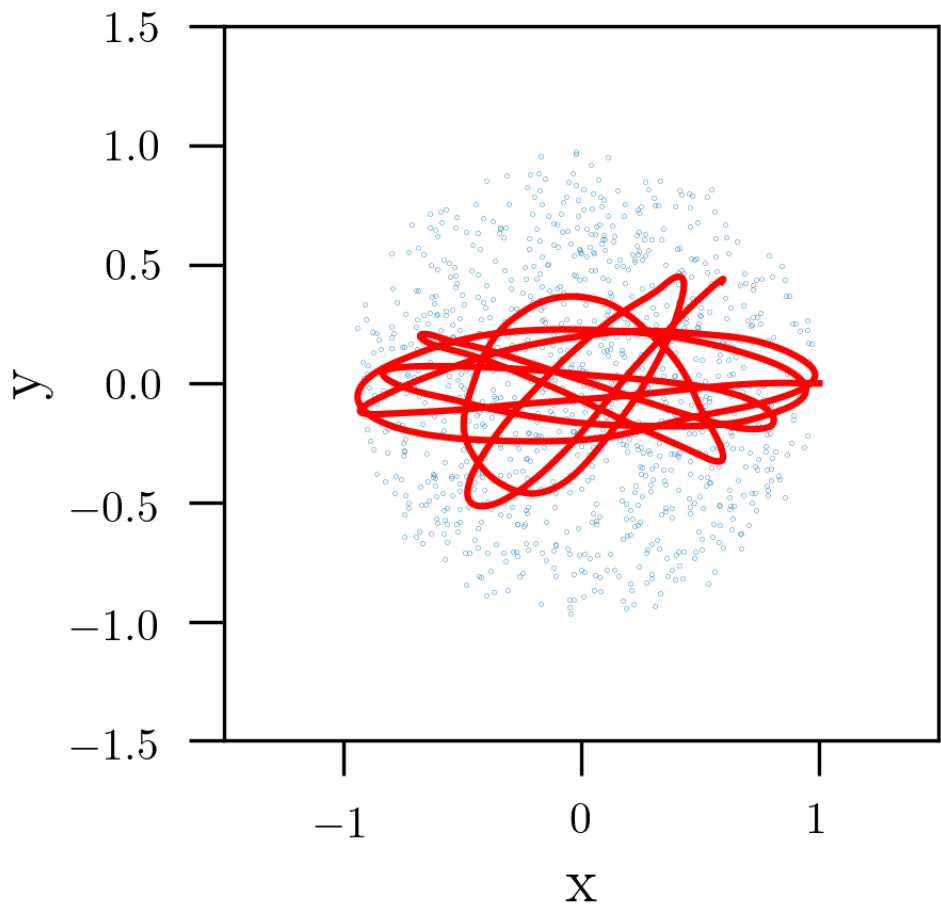
$t_{\text{relax}} = 20$

$N = 1000$ Time = 30.00



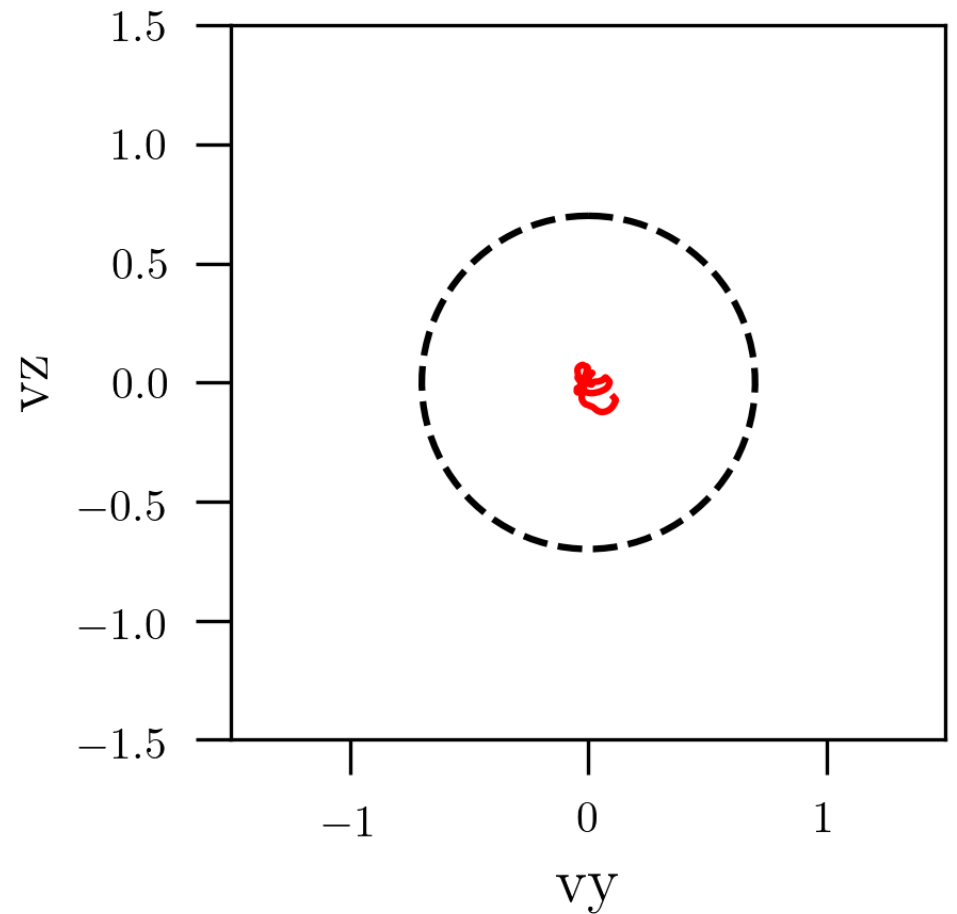
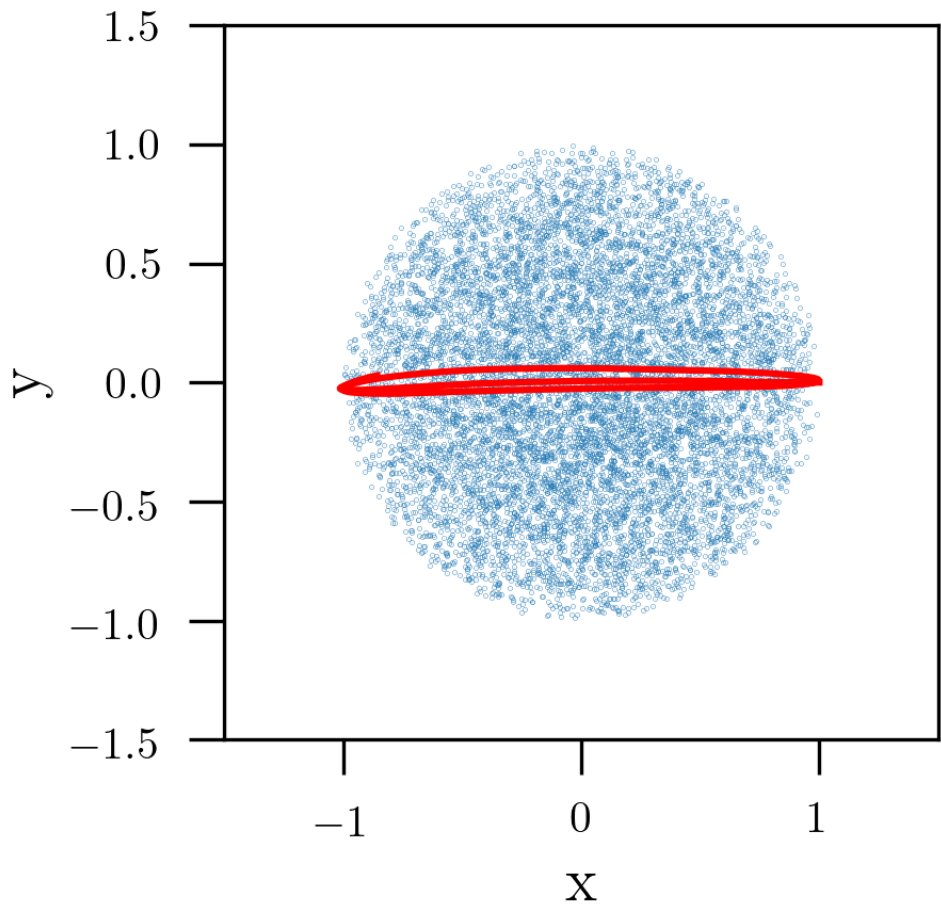
$t_{\text{relax}} = 20$

$N = 1000$ Time = 40.00



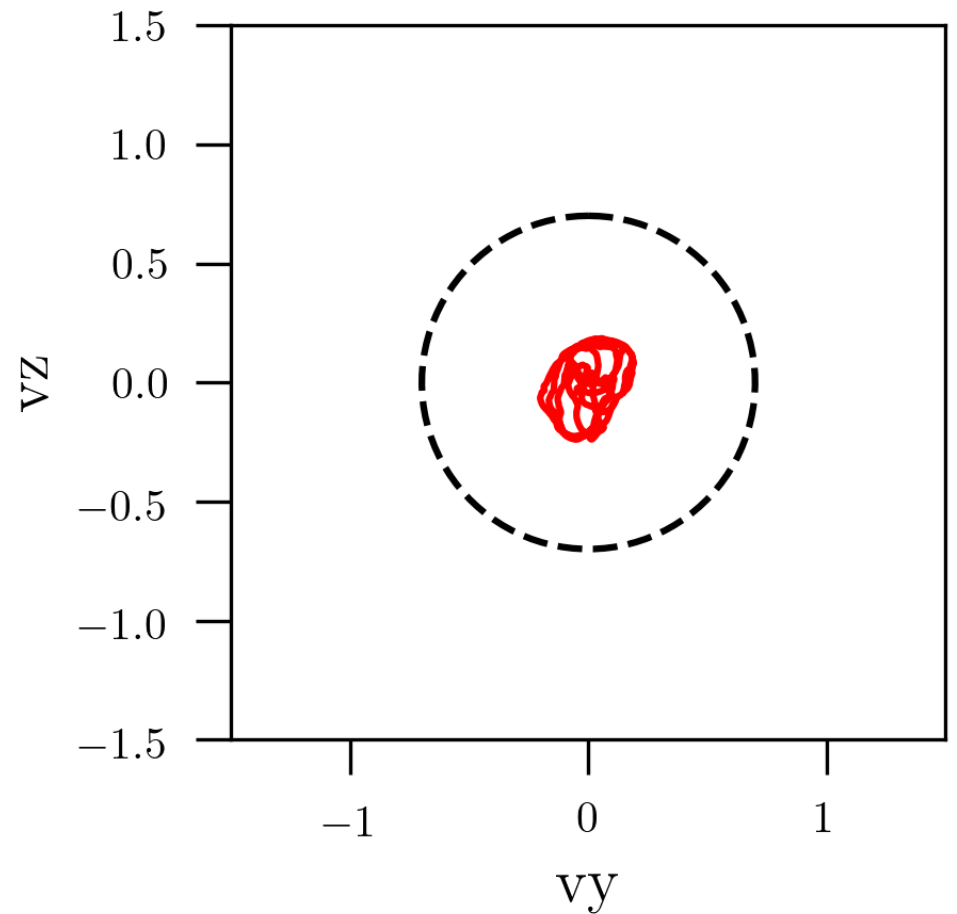
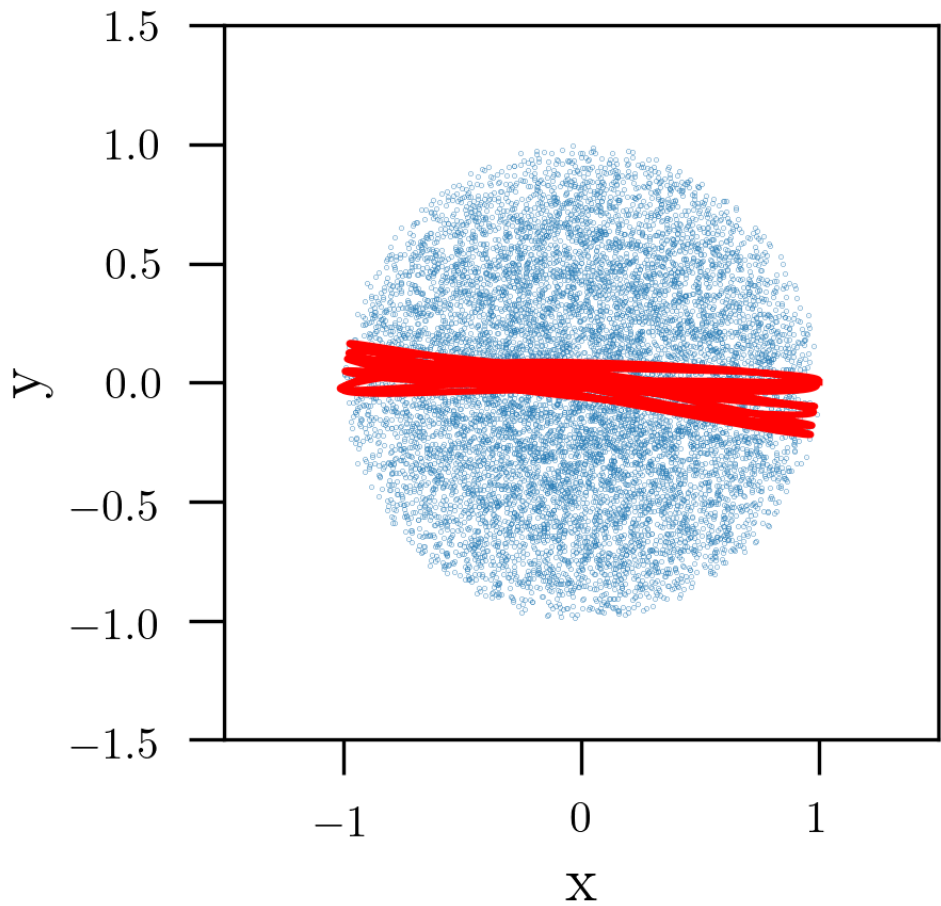
$t_{\text{relax}} = 20$

$N = 10000$ Time = 10.00



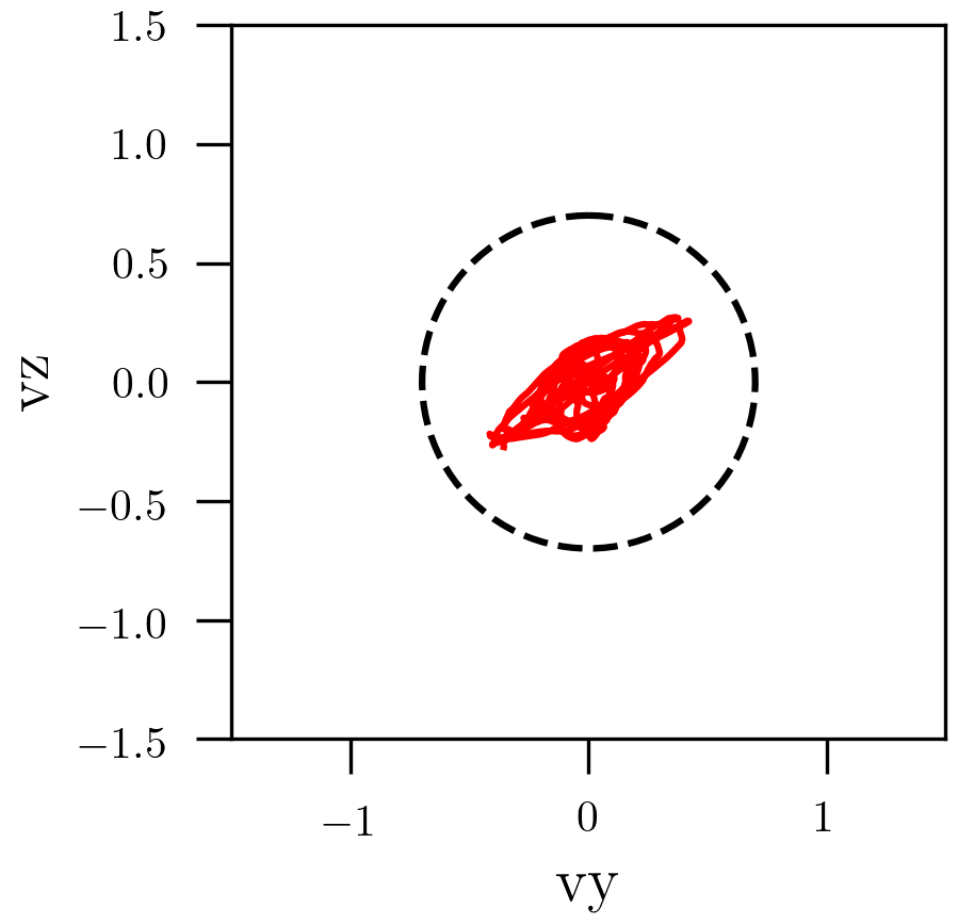
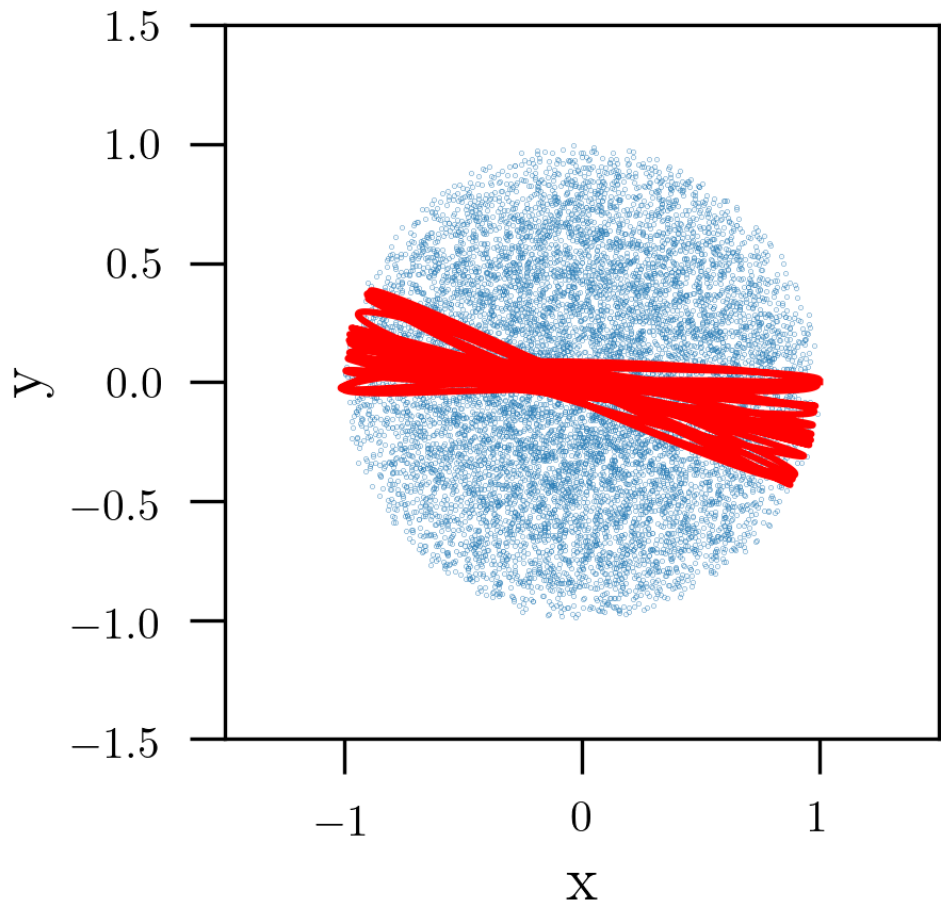
$t_{\text{relax}} = 150$

$N = 10000$ Time = 40.00



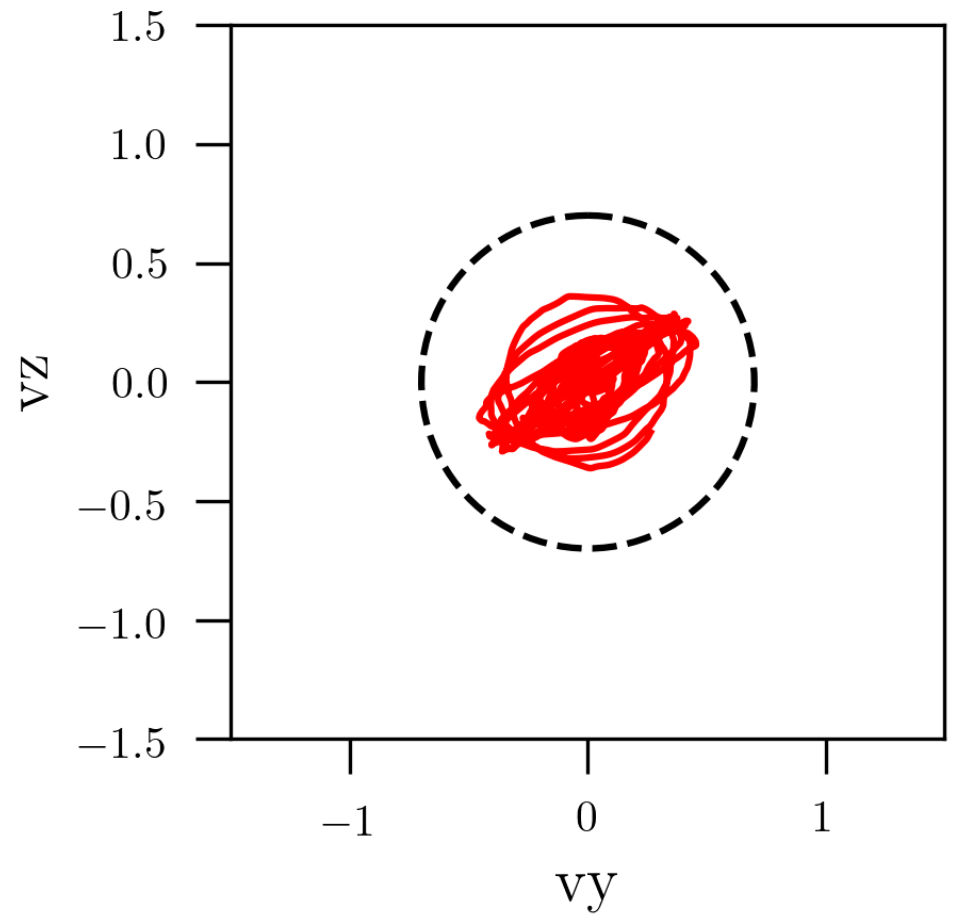
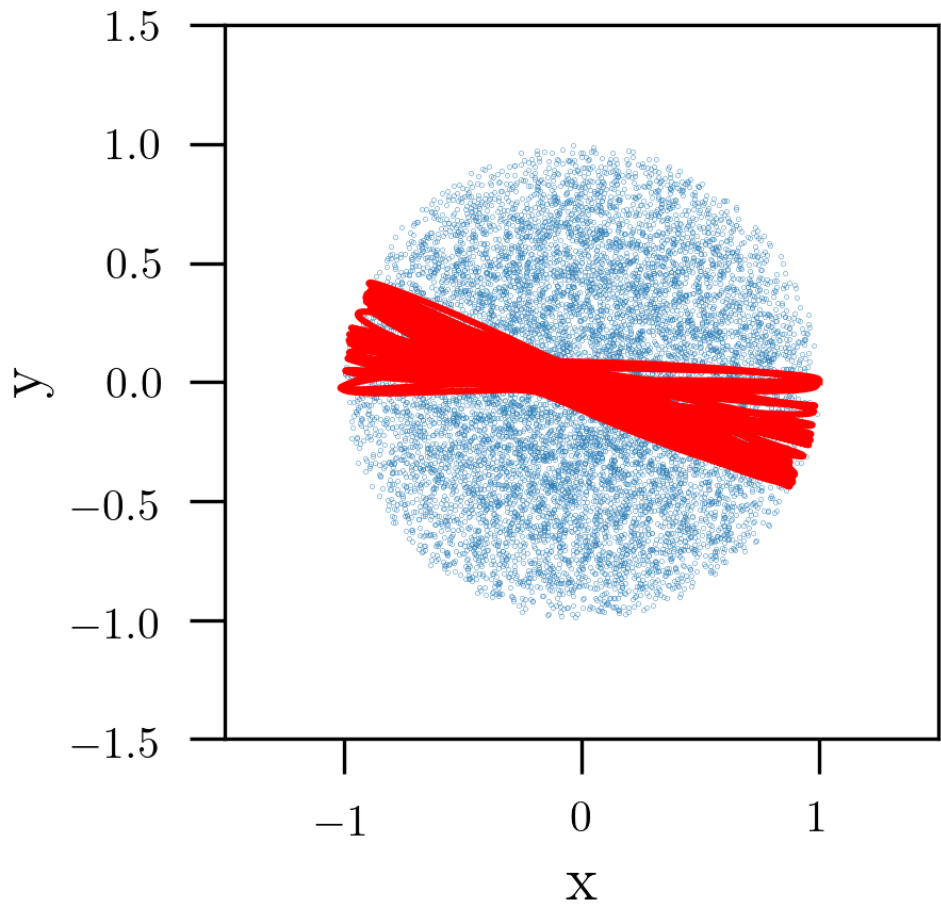
$t_{\text{relax}} = 150$

$N = 10000$ Time = 80.00



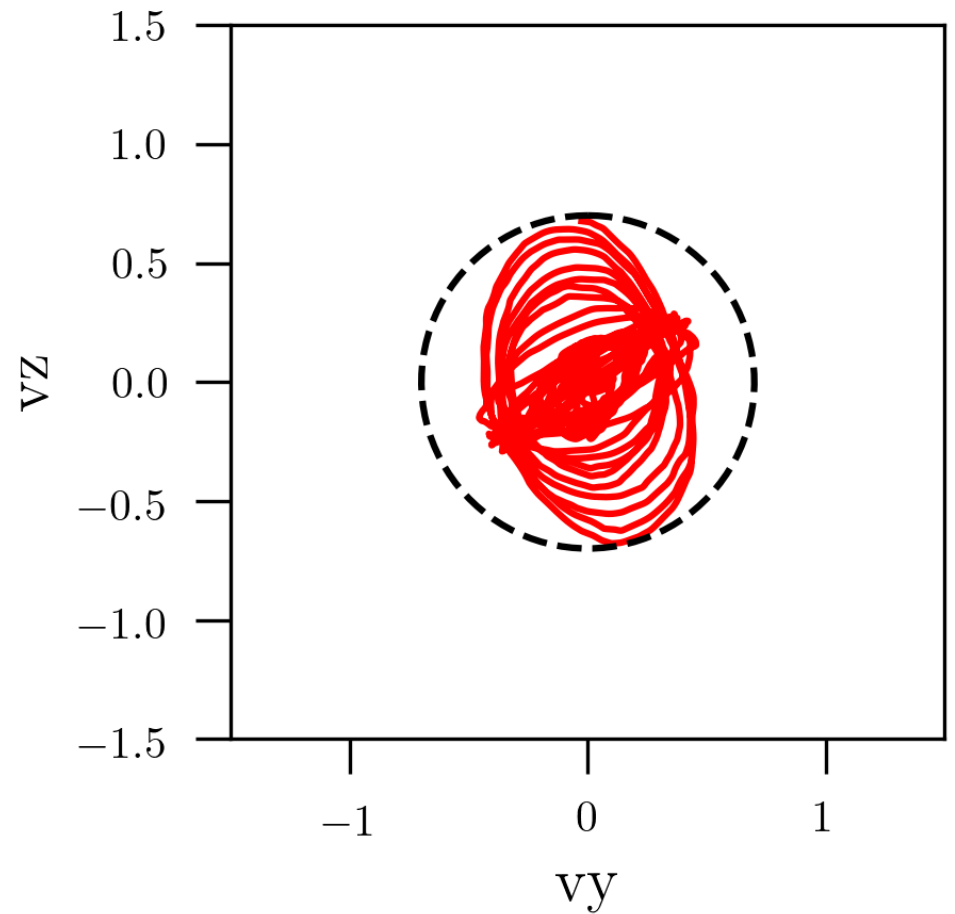
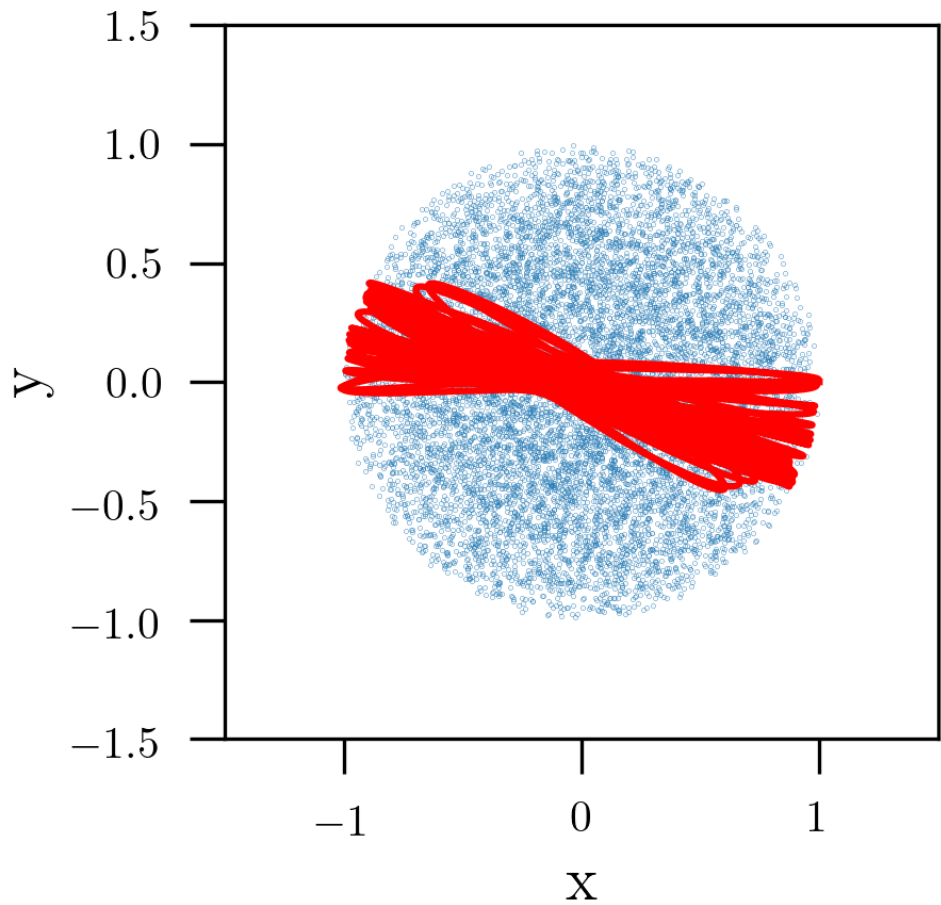
$t_{\text{relax}} = 150$

$N = 10000$ Time = 120.00



$t_{\text{relax}} = 150$

$N = 10000$ Time = 160.00



$t_{\text{relax}} = 150$

Potential Theory I

Potential theory : general results

Goal : compute the gravitational potential / forces
due to a large number of stars (point masses)

$N \sim 10^{11}$ for a Milky Way like galaxy

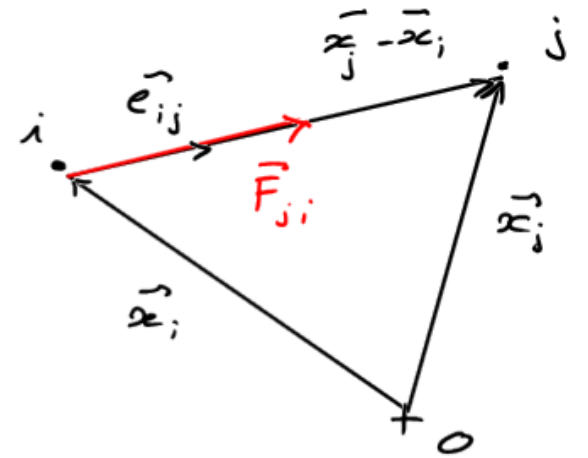
As the relaxation time of such system is very
large (\gg the age of the Universe) we can describe
the system with a smooth analytical potential / density.

Newton Law

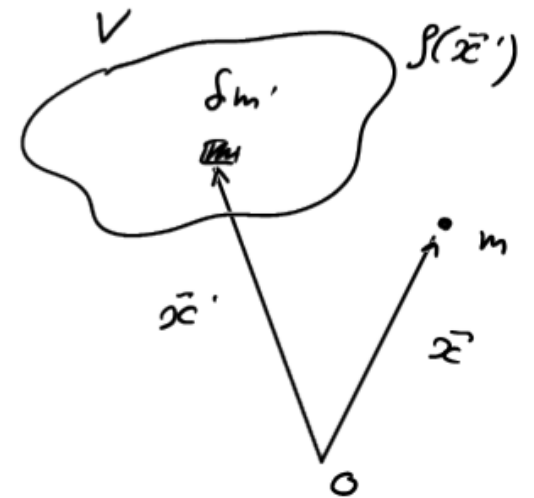
$$\vec{F}_{ji} = \frac{G m_i m_j}{|\vec{x}_j - \vec{x}_i|^2} \vec{e}_{ij} = \frac{G m_i m_j}{|\vec{x}_{ij}|^3} \vec{x}_{ij}$$

Force on a particle of mass m in \vec{x}
due to a distribution of mass $\rho(\vec{x})$

$$\begin{aligned} \delta \vec{F}(\vec{x}) &= \frac{G m \delta m'}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x}) \\ &= \frac{G m \rho(\vec{x}') d^3 \vec{x}'}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x}) \end{aligned}$$



$$\vec{x}_{ij} = \vec{x}_j - \vec{x}_i$$



So, the total force writes :

$$\vec{F}(\vec{x}) = \int_V \frac{G m \rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x}) d^3\vec{x}'$$

$$\equiv m \underbrace{G \int_V \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x}) d^3\vec{x}'}_{\vec{g}(\vec{x})}$$

$\vec{g}(\vec{x})$: gravitational field

$$[\vec{g}] = \frac{\text{cm}}{\text{s}^2} \equiv \frac{\text{erg}}{\text{g}} \frac{1}{\text{cm}}$$

Gravitational Potential

It is easy to see that the function

$$\delta V(\vec{x}) = - \frac{G m \delta m}{|\vec{x}' - \vec{x}|} \quad \text{is such that}$$

$$\vec{\nabla} \delta V(\vec{x}) = - \frac{G m \delta m}{|\vec{x}' - \vec{x}|^2} \frac{(\vec{x}' - \vec{x})}{|\vec{x}' - \vec{x}|} = - \delta \vec{F}(\vec{x})$$

so, by defining

$$V(\vec{x}) = - G \int_V \frac{m \rho(\vec{x}')}{|\vec{x}' - \vec{x}|} d^3\vec{x}'$$

we ensure that

$$\vec{\nabla} V(\vec{x}) = - \vec{F}(\vec{x})$$

We define the specific potential

$$\phi(\bar{x}) = \frac{V(\bar{x})}{m}$$

which writes

$$\phi(\bar{x}) = -G \int_V \frac{\rho(\bar{x}')}{|\bar{x}' - \bar{x}|} d^3\bar{x}'$$

$$[\phi] = \frac{\text{erg}}{g} \\ \equiv \text{specific energy}$$

The gravitational field writes:

$$\vec{g}(\bar{x}) = -\vec{\nabla} \phi(\bar{x})$$

Notes

- The gravity is a conservative force
- $\phi(\vec{x})$: scalar field
 $\vec{g}(\vec{x})$: vector field } contain the same information
- we will always use "specific" quantities

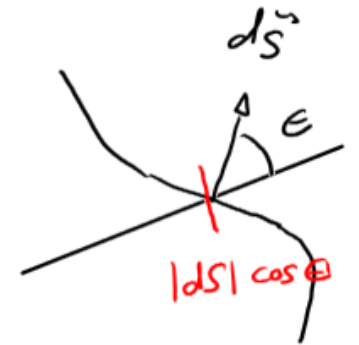
$$V(\vec{x}) \quad \rightarrow \quad \phi(\vec{x})$$

$$K = \frac{1}{2} m \vec{v}^2 \quad \rightarrow \quad \frac{1}{2} \vec{v}^2$$

$$\frac{1}{2} v^2 + \phi(\vec{x}) = \text{specific energy (conserved quantity)}$$

The Gauss's Law

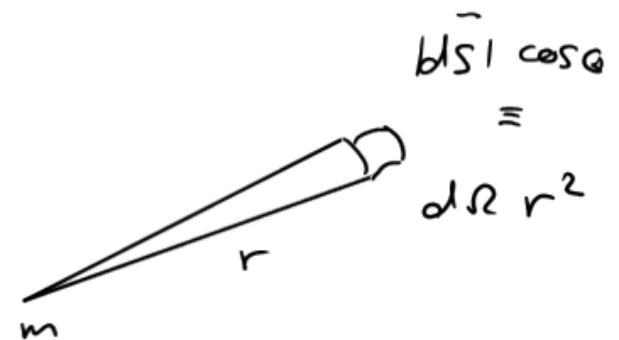
- Consider :
- a single point mass m
 - a surface S around this point
 - a point \vec{x} on the surface at a distance r
 - $\vec{g}(\vec{x})$ the gravitational field
 - $d\vec{S}$, the normal at the surface
 - θ the angle between $\vec{g}(\vec{x})$ and $d\vec{S}$



$$\vec{g}(\vec{x}) \cdot d\vec{S} = -|\vec{g}(\vec{x})| \cdot |d\vec{S}| \cos \theta$$

But $|d\vec{S}| \cos \theta = r^2 d\Omega$

$$|\vec{g}(\vec{x})| = \frac{Gm}{r^2}$$



$$\vec{g}(\vec{x}) \cdot d\vec{S} = -Gm d\Omega$$

integrating over any surface

$$\int_S \vec{g}(\vec{x}) \cdot d\vec{S} = \begin{cases} -4\pi G m \\ 0 \end{cases}$$

if m inside S
instead

For multiple masses m_i :

$$\int_S \vec{g}(\vec{x}) \cdot d\vec{S} = -4\pi G \sum_{i \in S} m_i$$

For a continuous mass distribution $\rho(\vec{x})$

$$\int_S \vec{g}(\vec{x}) \cdot d\vec{S} = -4\pi G \int_V \rho(\vec{x}) d\vec{x} = -4\pi G M$$

Gauss's Law

Divergence of the specific force

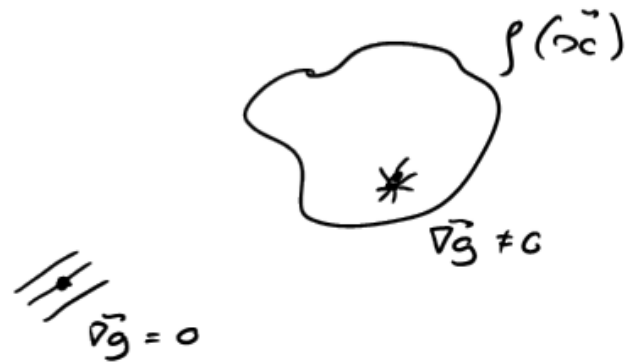
(A)

$$\vec{\nabla}_x \cdot \vec{g}(\vec{x})$$

$$\int_V \vec{\nabla} \cdot \vec{g}(\vec{x}) d^3\vec{x} \stackrel{\text{div. theorem}}{=} \int_S \vec{g}(\vec{x}) d\vec{S}$$

$$\stackrel{\text{Gauss's Law}}{=} -4\pi G \int_V \rho(\vec{x}) d\vec{x}$$

$$\boxed{\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = -4\pi G \rho(\vec{x})}$$



The Poisson Equation

$$\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = -4\pi G \rho(\vec{x})$$

with : $\vec{\nabla}_x \phi(\vec{x}) = -\vec{g}(\vec{x})$

$$\vec{\nabla}_x \cdot (\vec{\nabla}_x) = \vec{\nabla}_x^2$$

$$\vec{\nabla}_x^2 \phi(\vec{x}) = 4\pi G \rho(\vec{x})$$

Poisson Equation

Note : To ensure a unique solution, boundary conditions are necessary (2nd order diff. eqn.)

ex : $\phi(\infty) = 0$

$$\vec{\nabla} \phi(\infty) = \vec{g}(\infty) = 0$$

Divergence of the specific force

(B)

$$\vec{\nabla}_x \cdot \vec{g}(\vec{x})$$

$$\vec{g}(\vec{x}) = G \int_V \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x}) d^3\vec{x}'$$

$$\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = G \int_V \vec{\nabla}_x \cdot \left(\frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x}) \right) d^3\vec{x}'$$

$$\begin{aligned} \cdot \vec{\nabla}_x \cdot \left(\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) &= \frac{d}{dx_1} \left(\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) + \frac{d}{dx_2} \left(\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) + \frac{d}{dx_3} \left(\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) \\ &= -\frac{3}{|\vec{x}' - \vec{x}|^3} + \frac{3(\vec{x}' - \vec{x}) \cdot (\vec{x}' - \vec{x})}{|\vec{x}' - \vec{x}|^5} \\ &= \underline{\underline{0}} \quad \text{if} \quad \vec{x}' \neq \vec{x} \end{aligned}$$

$$\vec{\nabla}_{\vec{x}} \cdot \vec{g}(\vec{x}) = G \int_V \vec{\nabla}_{\vec{x}} \cdot \left(\frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x}) \right) d^3 \vec{x}'$$

$$= G \rho(\vec{x}) \int_{|\vec{x}' - \vec{x}| \leq h} \vec{\nabla}_{\vec{x}} \cdot \left(\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) d^3 \vec{x}'$$

$$= -G \rho(\vec{x}) \int_{|\vec{x}' - \vec{x}| \leq h} \vec{\nabla}_{\vec{x}'} \cdot \left(\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) d^3 \vec{x}'$$

$$= -G \rho(\vec{x}) \int_{|\vec{x}' - \vec{x}| = h} \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} d^2 S'$$

$$\underbrace{4\pi h^2 \cdot \frac{1}{r^2}}_{h=r} = 4\pi$$

$$\vec{\nabla}_{\vec{x}} \cdot \vec{g}(\vec{x}) = -4\pi G \rho(\vec{x})$$



variable exchange

$$\vec{\nabla}_{\vec{x}} \rho(\vec{x} - \vec{x}') = -\vec{\nabla}_{\vec{x}'} \rho(\vec{x} - \vec{x}')$$

divergence theorem

$$r = |\vec{x}' - \vec{x}| = h$$

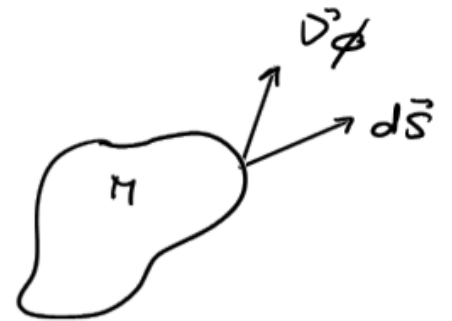
$$\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} = \frac{1}{r^2}$$

Gauss theorem (B) integrate the Poisson equation over a volume V that contains a mass M

$$\int_V \nabla^2 \phi(\vec{x}) d^3\vec{x} = \int_V 4\pi G \rho(\vec{x}) d^3\vec{x}$$

div.
Theorem

$$\int_S d^2\vec{s} \cdot \vec{\nabla} \phi = 4\pi G M$$



Gauss theorem

Equivalently :

$$\int_S d^2\vec{s} \cdot \vec{g}(\vec{x}) = -4\pi G M$$

Gauss's Law

Total potential energy (1.0)

Total work needed to assemble a density distribution $\rho(\vec{x})$



Assume a set of discrete points



• The work to bring the 1st point from ∞ to \vec{x}_1 is 0

• The work to bring the 2nd point from ∞ to \vec{x}_2 is $-\frac{Gm_1m_2}{r_{12}}$

• The work to bring the 3rd point from ∞ to \vec{x}_3 is $-\frac{Gm_1m_3}{r_{13}} - \frac{Gm_2m_3}{r_{23}}$

The total work is thus

$$W = -\frac{G m_1 m_2}{r_{12}} - \frac{G m_1 m_3}{r_{13}} - \frac{G m_2 m_3}{r_{23}} - \dots - \sum_{j=1}^{N-1} \frac{G m_{jN}}{r_{jN}}$$
$$= -\sum_{i=1}^N \sum_{j=1}^{i-1} \frac{G m_i m_j}{r_{ij}} = -\frac{1}{2} \sum_{i=1}^N \sum_{\substack{j=1 \\ i \neq j}}^N \frac{G m_i m_j}{r_{ij}}$$

With $\phi_i = -\sum_{\substack{j=1 \\ j \neq i}}^N \frac{G m_j}{r_{ij}}$ (potential on i)

$$W = \frac{1}{2} \sum_{i=1}^N m_i \phi_i = \frac{1}{2} \sum_{i=1}^N V_i$$

For a continuous mass distribution $\rho(\vec{x})$

$$W = \frac{1}{2} \int \rho(\vec{x}) \phi(\vec{x}) d^3 \vec{x}$$

Total potential energy (1.1)

From
$$W = \frac{1}{2} \int \rho(\vec{x}) \phi(\vec{x}) d^3\vec{x}$$

- replace $\rho(\vec{x})$ with the Poisson equation $\rho(\vec{x}) = \frac{1}{4\pi G} \nabla^2 \phi$

$$W = \frac{1}{8\pi G} \int \nabla^2 \phi \cdot \phi(\vec{x}) d^3\vec{x} = \frac{1}{8\pi G} \int \vec{\nabla} \cdot (\vec{\nabla} \phi) \cdot \phi(\vec{x}) d^3\vec{x}$$

- divergence theorem $\int d^3x \, g \cdot \vec{\nabla} \cdot \vec{F} = \int_S g \cdot \vec{F} d\vec{S} - \int d^3x \, \vec{F} \cdot \vec{\nabla} g$

$$W = \frac{1}{8\pi G} \left[\int \phi \vec{\nabla} \phi d\vec{S} - \int d^3\vec{x} \, \vec{\nabla} \phi \cdot \vec{\nabla} \phi \right]$$

$= 0$ as $\phi(\infty) = \vec{\nabla} \phi(\infty) = 0$

$$W = - \frac{1}{8\pi G} \int d^3\vec{x} \, |\vec{\nabla} \phi|^2$$

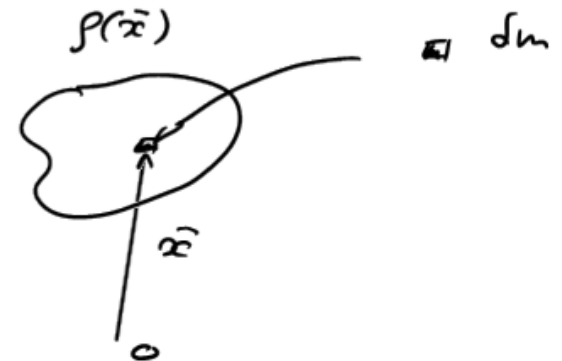
Total potential energy (2.0)

Total work needed to assemble a density distribution $\rho(\vec{x})$



- ① Work done to assemble a piece of mass $\delta m = \delta \rho d^3 \vec{x}$ from ∞ to \vec{x} assuming an existing mass distribution $\rho(\vec{x}), \phi(\vec{x})$

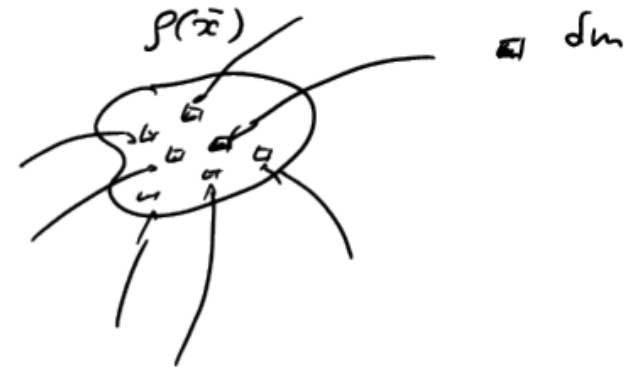
$$\begin{aligned} \delta W_{\vec{x}} &= V(\vec{x}) - \underbrace{V(\infty)}_{=0} \\ &= \delta m \phi(\vec{x}) = \delta \rho(\vec{x}) d^3 \vec{x} \phi(\vec{x}) \end{aligned}$$



To increase energy where the mass distribution by $\delta\rho$

$$\rho(\bar{x}) \rightarrow \rho(\bar{x}) + \delta\rho(\bar{x})$$

$$\delta W = \int \delta\rho(\bar{x}) d^3\bar{x} \phi(\bar{x})$$



Poisson:
$$\delta\rho(\bar{x}) = \frac{1}{4\pi G} \nabla^2 \delta\phi(\bar{x})$$

$$\delta W = \frac{1}{4\pi G} \int \nabla^2 \delta\phi(\bar{x}) \phi(\bar{x}) d^3\bar{x}$$

divergence theorem

$$\int_V d^3x \nabla \cdot \vec{F} = \int_S \vec{F} \cdot d^2s - \int_V d^3x \vec{F} \cdot \vec{\nu}$$

$$= \frac{1}{4\pi G} \underbrace{\int_{S \text{ at } \infty} \phi(\bar{x}) \nabla \delta\phi(\bar{x})}_{=0} - \frac{1}{4\pi G} \int \nabla \phi(\bar{x}) \cdot \nabla (\delta\phi(\bar{x})) d^3\bar{x}$$

as $\phi(\infty) = 0$

$$\nabla \delta\phi(\infty) = \delta g(\infty) = 0$$

$$\delta W = - \frac{1}{4\pi G} \int \vec{\nabla} \phi(\vec{x}) \cdot \vec{\nabla}(\delta \phi(\vec{x})) d^3 \vec{x}$$

with

$$\frac{1}{2} \delta |\vec{\nabla} \phi(\vec{x})|^2 = \delta \vec{\nabla} \phi(\vec{x}) \cdot \vec{\nabla} \phi(\vec{x}) = \vec{\nabla}(\delta \phi(\vec{x})) \cdot \vec{\nabla} \phi(\vec{x})$$

$$\delta W = - \frac{1}{8\pi G} \int \delta |\vec{\nabla} \phi|^2 d^3 x = - \frac{1}{8\pi G} \delta \int |\vec{\nabla} \phi|^2 d^3 x$$

② Contribution of all δW to W

$$W = - \frac{1}{8\pi G} \int |\vec{\nabla} \phi|^2 d^3 x$$

Total potential energy (2.2)

$$\text{From } W = -\frac{1}{8\pi G} \int |\vec{\nabla}\phi|^2 d^3x = -\frac{1}{8\pi G} \int \vec{\nabla}\phi \cdot \vec{\nabla}\phi d^3x$$

• divergence theorem $\int d^3x \vec{F} \cdot \vec{\nabla}g = \int_S g \cdot \vec{F} d\vec{S} - \int d^3x g \vec{\nabla} \cdot \vec{F}$

$$W = -\frac{1}{8\pi G} \left[\int_S \phi \vec{\nabla}\phi d\vec{S} - \int d^3x \phi \vec{\nabla}(\vec{\nabla}\phi) \right]$$

$= 0$ as $\phi(\infty) = \vec{\nabla}\phi(\infty) = 0$ $4\pi G \rho$ (Poisson)

$$= \frac{1}{8\pi G} 4\pi G \int d^3x \phi(\vec{x}) \rho(\vec{x})$$

$$W = \frac{1}{2} \int \rho(\vec{x}) \phi(\vec{x}) d^3x$$

Total potential energy : Summary

$$W = \frac{1}{2} \int \rho(\vec{x}) \phi(\vec{x}) d^3\vec{x}$$

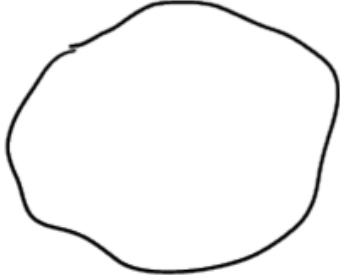
$$W = - \frac{1}{8\pi G} \int |\vec{\nabla} \phi|^2 d^3x$$

Other useful expression

$$W = - \int \rho(\vec{x}) \vec{x} \cdot \vec{\nabla} \phi(\vec{x}) d^3\vec{x}$$

Relation between the potential energy and the Poisson equation

What is the relation that must hold between the density $\rho(\vec{x})$ and potential $\phi(\vec{x})$ in order to minimize the potential energy of a system ?

$\rho(\vec{x})$
 $\phi(\vec{x})$  W : potential energy

Answer : the Poisson equation $\nabla^2 \phi = 4\pi G \rho$

The End