Final Exam

Exercise 1. Quiz. (12 points)

For each statement below, tell whether it is true or false (1 pt), and provide a justification if the answer is "true" / a counter-example if the answer is "false" (2 pts).

a) If a Markov chain is irreducible and recurrent, then it admits a stationary distribution.

b) If a Markov chain admits a unique stationary distribution, then it is irreducible.

c) Let P be the transition matrix of a finite and irreducible Markov chain with state space S, whose stationary distribution is uniform on S. Then $\sum_{i \in S} p_{ij} = 1$ for every $j \in S$.

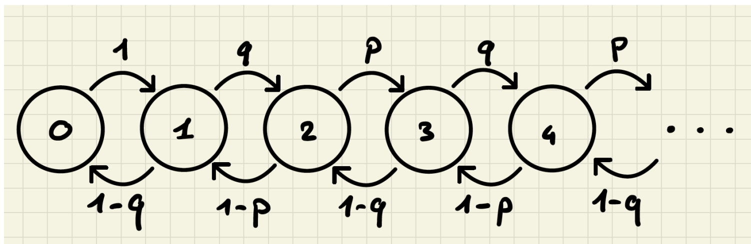
d) Let P be the transition matrix of a finite and irreducible Markov chain, whose stationary distribution satisfies moreover detailed balance. If λ is an eigenvalue of P such that $|\lambda| = 1$, then $\lambda = +1$.

Exercise 2. (20 points)

Let $0 < p, q < 1$ and $(X_n,\ n \geq 0)$ be the time-homogeneous Markov chain with state space $S = \mathbb{N}$, transition matrix P given by

$$\begin{cases} p_{0,1} = 1, \quad p_{2k,2k+1} = p = 1 - p_{2k,2k-1} \quad \text{for } k \geq 1 \\ p_{2k+1,2k+2} = q = 1 - p_{2k+1,2k} \quad \text{for } k \geq 0 \end{cases}$$

and the corresponding transition graph:



a) Describe the set of values of $0 < p, q < 1$ for which the chain $(X_n,\ n \geq 0)$ admits a stationary distribution π and compute this stationary distribution.

Hint: Try detailed balance !

b) Under the condition found in part a), is the chain $(X_n,\ n \geq 0)$ ergodic ? Justify.

c) For all values of $0 < p, q < 1$, compute $\mathbb{E}(T_0 | X_0 = 0)$, where $T_0 = \inf\{n \geq 1 \,:\, X_n = 0\}$.

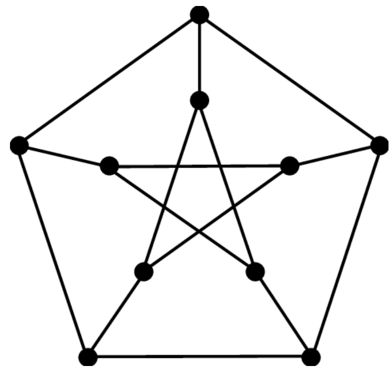
d) Let now $Q = P^2$. Explain in general why Q is guaranteed to be a transition matrix if P itself is a transition matrix.

e) Compute Q in the particular case of the present exercise.

f) Under the condition found in part a), is the Markov chain $(Y_n,\ n \geq 0)$ with transition matrix Q ergodic ? Justify.

Exercise 3. (16+3 points)

Consider the random walk on the Petersen (undirected) graph:



Let A be the adjacency matrix of this graph, defined as:

$$a_{ij} = \begin{cases} 1 & \text{if vertices } i \text{ and } j \text{ are connected by an edge} \\ 0 & \text{otherwise} \end{cases}$$

and let P be the transition matrix of the random walk on this graph, defined as

$$p_{ij} = \frac{a_{ij}}{d_i}, \quad \text{where } d_i \text{ is the degree of vertex } i$$

The aim of the present exercise is to compute the spectral gap of this random walk.

a) Explain why this random walk is irreducible and aperiodic (and therefore ergodic, as it is finite).

b) From a given vertex i, determine the set of all vertices j which are reachable in two steps or less with this random walk.

c) What is the stationary distribution π of the random walk ?

BONUS d) Show that $A^2 + A - 2I = J$, where I is the identity matrix and J is the "all ones" matrix, i.e., $J_{ij} = 1$ for all vertices i, j.

e) Using part d), deduce the set of possible values taken by the eigenvalues $\mu_0 \geq \mu_1 \geq \cdots \geq \mu_9$ of the adjacency matrix A.

Hint: The eigenvalues of the matrix J are given by

$$\nu_0 = 10, \quad \nu_1 = \nu_2 = \cdots = \nu_9 = 0$$

and watch out that only one eigenvalue μ_0 corresponds to the eigenvalue $\nu_0 = 10$.

f) Using part d), deduce the set of possible values taken by the eigenvalues $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_9$ of the transition matrix P.

g) Determine the value of the spectral gap γ of the random walk.

Exercise 4. (12 points)

Let $S = \mathbb{Z}^2$ and consider the following distribution on S:

$$\pi(i,j) = \frac{C}{1+i^2+j^2}, \quad (i,j) \in S$$

where $C > 0$ is the normalization constant such that $\sum_{(i,j)\in S} \pi(i,j) = 1$.

The aim of the present exercise is to sample from π using the Metropolis algorithm.

a) Which of the following base chains on S are appropriate to start with ? Justify your answers.

Remarks:

- We do *not* ask here that the base chain is aperiodic.

- Each of proposed base chains below is represented by its sole transition matrix ψ.

- Some drawings are clearly recommended here !

a1) $\psi_{(i,j),(k,l)} = \begin{cases} 1/4 & \text{if } k=i+1,\, l=j+1 \\ 1/4 & \text{if } k=i+1,\, l=j-1 \\ 1/4 & \text{if } k=i-1,\, l=j+1 \\ 1/4 & \text{if } k=i-1,\, l=j-1 \\ 0 & \text{otherwise} \end{cases}$

a2) $\psi_{(i,j),(k,l)} = \begin{cases} 1/2 & \text{if } k=i+1,\, l=j \\ 1/8 & \text{if } k=i-1,\, l=j \\ 1/4 & \text{if } k=i,\, l=j+1 \\ 1/8 & \text{if } k=i,\, l=j-1 \\ 0 & \text{otherwise} \end{cases}$

a3) $\psi_{(i,j),(k,l)} = \begin{cases} 1/4 & \text{if } k=i+1,\, l=j+1 \\ 1/4 & \text{if } k=i+1,\, l=j \\ 1/4 & \text{if } k=i,\, l=j+1 \\ 1/4 & \text{if } k=i-1,\, l=j-1 \\ 0 & \text{otherwise} \end{cases}$

a4) $\psi_{(i,j),(k,l)} = \begin{cases} 1/4 & \text{if } k=i+1,\, l=j+1 \\ 1/4 & \text{if } k=i+1,\, l=j \\ 1/4 & \text{if } k=i-1,\, l=j \\ 1/4 & \text{if } k=i-1,\, l=j-1 \\ 0 & \text{otherwise} \end{cases}$

b) Consider now the base chain whose transition matrix is given by

$$\psi_{(i,j),(k,l)} = \begin{cases} 1/4 & \text{if } k=i+1,\, l=j \\ 1/4 & \text{if } k=i-1,\, l=j \\ 1/4 & \text{if } k=i,\, l=j+1 \\ 1/4 & \text{if } k=i,\, l=j-1 \\ 0 & \text{otherwise} \end{cases}$$

and compute the acceptance probabilities $a_{(i,j),(k,l)}$ of the Metropolis chain.

Remark: You may restrict yourselves to states in the first quadrant $\{(i,j) \in S \,:\, i \geq 0,\, j \geq 0\}$).

c) Among all possible moves $(i,j) \to (k,l)$ proposed by the base chain ψ (that from part b), which have the least acceptance probability ? (multiple answers are possible, but only one is required)