Final Exam – SG0211

This exam is open book. No electronic devices of any kind are allowed. There are 4 problems. Choose the ones you find easiest and collect as many points as possible. We do not necessarily expect you to finish all of them. Good luck!

Name: _____________________________

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<th>Problem</th>
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<td>Problem 1</td>
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<td>Problem 4</td>
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<td><strong>Total</strong></td>
<td><strong>/ 62</strong></td>
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Problem 1 (Fisher Goes Exponential). [15 pts]

Let $p_\theta(x)$ denote a family of distributions parameterized by $\theta$. Define the Fisher information as

$$I_\theta = \mathbb{E}_\theta[\nabla_\theta \log p_\theta(X)(\nabla_\theta \log p_\theta(X))^T].$$

(1) [5pts] Let $p_\theta(x) = h(x)e^{\langle \theta, \phi(x) \rangle - A(\theta)}$ be an exponential family. What is the Fisher information in terms of the parameters of the family?

(2) [5pts - 1pt per question] Consider distributions of the form $p_\lambda(x) = \lambda e^{-\lambda x}$, where $\lambda \in \mathbb{R}^+$. 
1. Write it in the form of an exponential family.
2. What is $\Theta = \{\theta \in \mathbb{R} : A(\theta) < \infty\}$.
3. Is the family regular?
4. Is it minimal?
5. What is the Fisher information?

(3) [5pts - 1pt per question] Consider distributions of the form $p_p(k) = (1 - p)^k p$, where $p \in (0, 1)$ and $k \in \mathbb{N}$.
1. Write it in the form of an exponential family.
2. What is $\Theta = \{\theta \in \mathbb{R} : A(\theta) < \infty\}$.
3. Is the family regular?
4. Is it minimal?
5. What is the Fisher information?
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Problem 2 (Compression). [15 pts]

Suppose \( \mathcal{P} \in \Pi(\mathcal{X}, \mathcal{Y}) \) be a probability distribution on \( \mathcal{X} \times \mathcal{Y} \) and \( (X, Y) \) be a joint random variable with distribution \( P_{XY} \) with marginals \( P_X \) and \( P_Y \).

In what follows, assume that all codes are optimal, prefix-free, and binary. Optimal here means having smallest possible average length. All logs are to the base 2.

(1) [1 pt] Let \( c_X : \mathcal{X} \to \{0, 1\}^* \) and \( c_Y : \mathcal{Y} \to \{0, 1\}^* \) be optimal prefix free codes. What are lower and upper bounds for the expected length of these codes \( c_X \) and \( c_Y \)?

(2) [1 pt] Let \( c_{XY} : \mathcal{X} \times \mathcal{Y} \to \{0, 1\}^* \) be an optimal prefix free code. What are lower and upper bounds for the expected length of this code?

(3) [10 pts total] In this sub problem, assume that \( X, Y \) have a joint distribution according to the following table:

<table>
<thead>
<tr>
<th>X</th>
<th>Y=0</th>
<th>Y=1</th>
</tr>
</thead>
<tbody>
<tr>
<td>X=0</td>
<td>( \frac{1}{4} )</td>
<td>0</td>
</tr>
<tr>
<td>X=1</td>
<td>( \frac{1}{8} )</td>
<td>( \frac{1}{8} )</td>
</tr>
<tr>
<td>X=2</td>
<td>( \frac{1}{8} )</td>
<td>( \frac{1}{8} )</td>
</tr>
<tr>
<td>X=3</td>
<td>0</td>
<td>( \frac{1}{4} )</td>
</tr>
</tbody>
</table>

(a) [4 pts] What are lower and upper bounds for the expected lengths of \( c_X \) and \( c_Y \)? Are the lower bounds tight?

(b) [3 pts] What are lower and upper bounds for the expected lengths of \( c_{XY} \)? Is the lower bound tight?

(c) [3 pts] For the above joint distribution, is it more efficient to compress separately and concatenate the individual code words (which, as we saw in the lecture, is guaranteed to yield a prefix free code), or to compress \( (X, Y) \) jointly (again, in a prefix free manner)?

(4) [3 pts] Assume that \( (X, Y) \) has some generic joint distribution. Assume further that \( I(X; Y) > 1 \). Show that in this case optimal joint prefix free compression is more efficient than compressing individually and concatenating.
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Problem 3 (Stability implies Generalization). [12 pts]

Let \( S = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\} \) be a training dataset composed of \( n \) i.i.d. samples drawn from \( D \). As usual, we denote \( L_D(h) = E_{(x,y) \sim D}[l(h(x), y)] \) and \( L_S(h) = \frac{1}{n} \sum_{i=1}^{n} l(h(x_i), y_i) \) the true and empirical risks of a hypothesis \( h \), respectively. For simplicity, let us denote by \( h_S \) the output of a learning algorithm when trained with dataset \( S \).

An important property of learning algorithms is their ability to generalize, i.e., the true and empirical risks of the output hypothesis should be close in expectation. Formally, we say that a learning algorithm \( \mathcal{A} \) \( \epsilon \)-generalizes in expectation if

\[
|E_S[L_S(h_S)] - L_D(h_S)| < \epsilon .
\]  

(1)

An interesting connection arises when we investigate the stability of a learning algorithm. Formally, we call a learning algorithm \( \epsilon \)-uniformly stable if \( \forall S, S' \) datasets of size \( n \) that differ in at most one sample we have

\[
\sup_{(x,y)} l(h_S(x), y) - l(h_{S'}(x), y) < \epsilon .
\]  

(2)

Notations: \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n), (\bar{x}_1, \bar{y}_1), \ldots, (\bar{x}_n, \bar{y}_n)\) are \( 2n \) independently sampled training examples. We define \( S = \{(x_1, y_1), \ldots, (x_n, y_n)\}, \tilde{S} = \{(\bar{x}_1, \bar{y}_1), \ldots, (\bar{x}_n, \bar{y}_n)\} \) and \( S^{(i)} = \{(x_1, y_1), \ldots, (x_{i-1}, y_{i-1}), (\bar{x}_i, \bar{y}_i), (x_{i+1}, y_{i+1}), \ldots, (x_n, y_n)\} \).

(1) [2 pts] Prove that \( L_D(h_S) = E_S[\frac{1}{n} \sum_{i=1}^{n} l(h_S(x_i), y_i)]. \)

(2) [3 pts] Prove that \( E_{S,\tilde{S}}[l(h_S(\bar{x}_i), \bar{y}_i)] = E_{S,S^{(i)}}[l(h_{S^{(i)}}(x_i), y_i)]. \)
(3) [7 pts] Prove that an $\epsilon$-uniformly stable learning algorithm $\epsilon$-generalizes in expectation, by justifying each step in the following sequence.

$$
|E_{S}[L_{S}(h_{S}) - L_{D}(h_{S})]| \overset{(a)}{=} |E_{S} \left[ L_{S}(h_{S}) - E_{\tilde{S}} \left[ \frac{1}{n} \sum_{i=1}^{n} l(h_{S}(\tilde{x}_i), \tilde{y}_i) \right] \right] |
$$

$$
\overset{(b)}{=} |E_{S} [L_{S}(h_{S})] - E_{S,\tilde{S}} \left[ \frac{1}{n} \sum_{i=1}^{n} l(h_{S}(\tilde{x}_i), \tilde{y}_i) \right] |
$$

$$
\overset{(c)}{=} |E_{S} [L_{S}(h_{S})] - \frac{1}{n} \sum_{i=1}^{n} E_{S,\tilde{S}} [l(h_{S}(\tilde{x}_i), \tilde{y}_i)] |
$$

$$
\overset{(d)}{=} |E_{S} [L_{S}(h_{S})] - \frac{1}{n} \sum_{i=1}^{n} E_{S,(x_i,y_i)} [l(h_{S}(x_i), y_i)] |
$$

$$
\overset{(e)}{=} |E_{S} \left[ \frac{1}{n} \sum_{i=1}^{n} l(h_{S}(x_i), y_i) \right] - \frac{1}{n} \sum_{i=1}^{n} E_{S,(x_i,y_i)} [l(h_{S}(x_i), y_i)] |
$$

$$
\overset{(f)}{=} \left| \frac{1}{n} \sum_{i=1}^{n} E_{S,(x_i,y_i)} [l(h_{S}(x_i), y_i)] - l(h_{S}(x_i), y_i) \right| |
$$

$$
\overset{(g)}{\leq} \frac{1}{n} \sum_{i=1}^{n} \epsilon = \epsilon
$$
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We consider the following game where in each round $t$ we can choose between $[N] = \{1, 2, \ldots, N\}$ different actions. After we choose an action $a_t \in [N]$ an adversary reveals the loss of each action in this round, call it $l^i_t \in [0, 1]$, $i \in [N]$. Note that this is an adversarial setting, where the losses do not come from a probability distribution. This setting differs from what we had discussed in class where only the loss for the chosen action was revealed.

Our goal is to design a randomized algorithm $A$ which maintains a probability distribution $p^t$ over actions, and achieves a sub-linear regret, i.e., $R(T) = \max_i \{\sum_{t=1}^T E_{A_t \sim p^t} [l^i_{A_t} - l^i_t]\} \leq o(T)$. We also note that the adversary may know the probability distribution $p^t$, but does not know the realizations $A_t$. We will analyze the following algorithm:

**Algorithm 1: Multiplicative Weights Update**

**Input:** learning parameter $\epsilon$

**Initialization:** $p^1_i = 1/N$, $w^1_i = 1$, $\forall i \in [N]$, $\Phi^1 = N$

for $t = 1$ to $T$

$A_t \sim p^t$

Adversary reveals the loss vector $l^t$ and we suffer $l^t_{A_t}$

Update weights $w^{t+1}_i = w^t_i \cdot \exp(-\epsilon \cdot l^t_i)$, $\forall i \in [N]$ and let $\Phi^{t+1} = \sum_i w^{t+1}_i$

Update the probability distribution: $p^{t+1}_i = w^{t+1}_i / \Phi^{t+1}$, $\forall i$

end for

(1) [2 pts] Prove that $w^{T+1}_i = \exp(-\epsilon \cdot \sum_{t=1}^T l^t_i)$, $\forall i \in [N]$ 

(2) [8 pts] Prove that $\Phi^{t+1} \leq \Phi^t \cdot \exp(\epsilon^2 - \epsilon \langle p^t, l^t \rangle)$

*Hint:* Note that $w^{t+1}_i = p^{t+1}_i \cdot \Phi^{t+1}$ and use the inequalities: (a) $e^x \leq 1 + x + x^2$, $\forall x \in [0, 1]$ and (b) $e^x \geq x + 1$, $\forall x$.

(3) [2 pts] Prove that $\Phi^{T+1} \leq \Phi^1 \cdot \exp(\epsilon^2 \cdot T - \epsilon \sum_{t=1}^T \langle p^t, l^t \rangle)$

(4) [8 pts] By noting that $\Phi^1 \cdot \exp(\epsilon^2 \cdot T - \epsilon \sum_{t=1}^T \langle p^t, l^t \rangle) \geq \Phi^{T+1} \geq w^{T+1}_i$, $\forall i \in [N]$ set the learning parameter $\epsilon$ so that $R(T) \leq 2\sqrt{\log(N) \cdot T}$. 

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