Quantum Information Processing

Final exam
Fall term 2022

Assignment date: February 1, 2023, 15h15
Due date: February 1, 2023, 18h15

COM 309 – Exam – room CE 4

- There are 3 problems: write your solutions in the indicated space.
- No electronic devices are allowed.
- Dont forget to write your name below.
- Good luck!

Name: ____________________________

Section: ______________

Sciper No.: ______________

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Useful identities

- For all $z \in \mathbb{C}^*$, you can write $z = |z|e^{i \arg z}$
- The moment generating function of a gaussian distribution $X \sim \mathcal{N}(\mu, \sigma^2)$ is:

$$\mathbb{E}[e^{tX}] = e^{\mu t + \frac{1}{2}\sigma^2 t^2} \tag{1}$$

- We define the Hadamard basis

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{2}$$

- We define the Pauli matrices:

$$\sigma_x = X = |0\rangle \langle 1| + |1\rangle \langle 0| \tag{3}$$
$$\sigma_z = Z = |0\rangle \langle 0| - |1\rangle \langle 1| \tag{4}$$
$$\sigma_y = Y = iXZ \tag{5}$$

- We recall the following formula for any *unitary* vector $\vec{n}$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and I the identity matrix:

$$e^{i\alpha \vec{n}\cdot\vec{\sigma}} = \cos(\alpha)I + i\sin(\alpha)\vec{n}\cdot\vec{\sigma} \tag{6}$$

- Depending on the context, we use: $|\uparrow\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|\downarrow\rangle = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Problem 1. (10 points) *A dense-coding protocol with a third-party*

In this problem, we will revisit the **dense-coding protocol** between Alice and Bob but with an additional third-party: Charlie. The protocol works as follow:

1. Charlie is responsible for generating an entangled state:

$$|GHZ\rangle = \frac{1}{\sqrt{2}} \left(|000\rangle_{ABC} + |111\rangle_{ABC} \right)$$

   and distributes one qubit for Alice (A), one for Bob (B) and keeps one for himself (C).

2. Alice wants to send a message m of 2 classical bits. To this end, say $m = b_1 b_2$ where b_1 and b_2 are the two respective bits, she transforms her qubit with the operator $U = Z^{b_1} X^{b_2}$ and sends it to Bob.

3. Bob receives the qubit from Alice and make a measurement in the orthonormal basis $\{|\beta_{00}\rangle, |\beta_{10}\rangle, |\beta_{01}\rangle, |\beta_{11}\rangle\}$ given by:

$$|\beta_{00}\rangle = \left(\frac{|00\rangle_{AB} + |11\rangle_{AB}}{\sqrt{2}} \right), \qquad |\beta_{ij}\rangle = \left(Z^i X^j \otimes I \right) |\beta_{00}\rangle$$

(a) (1 point) What are the possible outcomes for Bob from his measurement?

(b) (3 points) For each $(i, j) \in \{(1, 0), (0, 1), (1, 1)\}$, express the value of $|\beta_{ij}\rangle$ in the computational basis.

(c) (3 points) Say Alice wants to send $m = 10$. What is the global state of the system after Alice's transformation and before Bob's measurement? Calculate the probability of the outcome $|\beta_{10}\rangle$ for Bob. Is he able to reconstruct the message m from Alice as seen in the dense coding protocol?

(d) (1 point) We will now see how Charlie can give a "key" to Bob in order for him to fully reconstruct the message of Alice. First of all, show that we have:

$$|GHZ\rangle = \frac{1}{\sqrt{2}} \left(|\beta_{00}\rangle \otimes |+\rangle + |\beta_{10}\rangle \otimes |-\rangle \right)$$

(e) (1 point) Assume now that Charlie makes a measurement on his qubit in the orthonormal basis $\{|+\rangle, |-\rangle\}$. Assume further that the outcome is $|+\rangle$. If Alice still wants to send $m = 10$, what is the probability of obtaining $|\beta_{10}\rangle$ for Bob?

(f) (1 point) Assume now that the qubit of Charlie collapsed to $|-\rangle$. What are the possible outcomes and their probabilities for Bob?

Problem 2. (10 points) *Spin dynamics: Ramsey sequence of operations*

Consider the Hamiltonian

$$H = \frac{\hbar\delta}{2}\sigma_z - \frac{\hbar\omega_1}{2}\sigma_x$$

Recall that in class we encountered this Hamiltonian as the one of a spin in a static along the z axis + rotating magnetic field in the (xy) plane. Here $\delta = \omega - \omega_0$ is the detuning parameter, between ω_0 the Larmor frequency and ω the frequency of the rotating field, whereas ω_1 is the strength of the rotating field.

But this Hamiltonian also models qubits or two energy levels of atoms in suitable regimes.

In this problem we consider the so-called *Ramsey sequence of operations*:

- A $\frac{\pi}{2}$ pulse: this is a time evolution during the time interval $[0, \tau]$ with $\tau = \frac{\pi}{2\omega_1}$ and $\delta = 0$.
- A Larmor precession during the time interval $[\tau, \tau + T]$ with $\omega_1 = 0$ and $\delta > 0$.
- A $\frac{\pi}{2}$ pulse as before during the time interval $[\tau + T, 2\tau + T]$.

We assume that the initial state of the spin is $|\uparrow\rangle$.

(a) (3 points) Compute the state at times τ, $\tau + T$, $2\tau + T$. *Hint*: we recall the formula for the time evolution operator $U_t = \exp(-i\frac{tH}{\hbar})$

(b) (3 points) Compute the probabilities that at the final time the spin is observed in states $|\uparrow\rangle$ or $|\downarrow\rangle$. Plot the probability $\mathbb{P}(|\uparrow\rangle_{t=0} \to |\downarrow\rangle_{t=2\tau+T})$ as function of T.

(c) (3 points) Illustrate the two trajectories of the spin on the Bloch spheres for $T = \frac{\pi}{\delta}$ and $T = \frac{2\pi}{\delta}$ and describe them in a few words as well.

(d) (1 point) Can you describe an analogy between the Ramsey sequence of operations and the Mach-Zehnder interferometer seen in class ?

Problem 3. (12 points) *Density matrix: a decoherence model*

In the following, we will study a model of decoherence of one qubit interacting with the environment. The whole system is defined in the hilbert space $\mathcal{H} = \mathcal{H}_{\mathcal{E}} \otimes \mathcal{H}_b$ where $\mathcal{H}_{\mathcal{E}}$ is the Hilbert space describing the possible states of the environment and $\mathcal{H}_b = \mathbb{C}^2$ is the Hilbert space describing the possible states of the qubit.

Let $|\phi_0\rangle = \alpha |0\rangle + \beta |1\rangle \in \mathcal{H}_b$ be the initial state of the qubit and $|\mathcal{E}\rangle \in \mathcal{H}_b$ that of the environment (or sometimes called *heat-bath*).

(a) (1 point) What is the initial global state $|\psi_0\rangle$ of the whole system?

Let $(|i\rangle)_{i\geq 1} \in \mathcal{H}_{\mathcal{E}}$ be an "infinite" orthonormal basis of the environment $\mathcal{H}_{\mathcal{E}}$. We define the evolution operator $U = \sum_{i=1}^{+\infty} |i\rangle\langle i| \otimes \mathcal{D}(\theta_i)$ for some distinct angles $\theta_i \in \mathbb{R}$, and the dephasing operator: $\mathcal{D}(\theta_i) = |0\rangle\langle 0| + e^{i\theta_i} |1\rangle\langle 1|$.

If the environment makes a transition from state $|\mathcal{E}\rangle$ to $|i\rangle$, we let $\mu(\theta_i) = P(|\mathcal{E}\rangle \to |i\rangle)$ the probability of such a transition.

(b) (2 points) Show that U is a unitary operator (describe your steps).

(c) (4 points) The state of the system evolves with a power $n \in \mathbb{N}$ of the operator U as $|\psi_n\rangle = U^n |\psi_0\rangle$. Show that $\mathcal{D}(\theta_i)^n = \mathcal{D}(n\theta_i)$ and deduce that

$$|\psi_n\rangle = \sum_{i=1}^{+\infty} e^{i \arg\langle i|\mathcal{E}\rangle} \sqrt{\mu(\theta_i)}\, |i\rangle \otimes (\mathcal{D}(n\theta_i)\, |\phi_0\rangle)$$

(d) (1 point) Now let's consider the density matrix of the qubit itself: $\rho_n = \mathrm{Tr}_{\mathcal{H}_{\mathcal{E}}}[|\psi_n\rangle\langle\psi_n|]$. First, using only the result of question (a), show that we have initially:

$$\rho_0 = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}$$

And give its Von Neumann entropy S_0.

(e) (1 point) For any angle $\theta \in \mathbb{R}$, show that we have:

$$\mathcal{D}(\theta)\rho_0\mathcal{D}(\theta)^\dagger = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* e^{-i\theta} \\ \alpha^*\beta e^{i\theta} & |\beta|^2 \end{pmatrix}$$

(f) (2 points) Now let's consider $\hat{\theta}$ a random variable in $\mathbb{R}$ with partial distribution function (PDF) $\theta \to \mu(\theta)$. Use the result of question (c) and (e) to show that the density matrix of the qubit coincide with the following expression:

$$\rho_n = \begin{pmatrix} |\alpha|^2 & \alpha\beta^*\mathbb{E}[e^{-in\hat{\theta}}] \\ \alpha^*\beta\mathbb{E}[e^{in\hat{\theta}}] & |\beta|^2 \end{pmatrix}$$

(g) (1 point) Now say that μ is the PDF of a gaussian distribution of mean 0 and variance σ^2. Show that the density matrix of the qubit evolves as:

$$\rho_n = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* e^{-\frac{1}{2}\sigma^2 n^2} \\ \alpha^*\beta e^{-\frac{1}{2}\sigma^2 n^2} & |\beta|^2 \end{pmatrix}$$

Calculate $\rho_\infty = \lim_{n\to\infty} \rho_n$ and give the associated entropy S_∞. Compare it with S_0 found in (d) and comment on the result.