Quantum Information Processing

Final exam	Assignment date: February 1, 2023, 15h15
Fall term 2022	Due date: February 1, 2023, 18h15

COM 309 – Exam – room CE 4

- There are 3 problems: write your solutions in the indicated space.
- No electronic devices are allowed.
- Dont forget to write your name below.
- Good luck!

Name: ____________________________

Section: ______________

Sciper No.: ______________

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Useful identities

- For all $z \in \mathbb{C}^*$, you can write $z = |z|e^{i \arg z}$
- The moment generating function of a gaussian distribution $X \sim \mathcal{N}(\mu, \sigma^2)$ is:

$$\mathbb{E}[e^{tX}] = e^{\mu t + \frac{1}{2}\sigma^2 t^2} \tag{1}$$

- We define the Hadamard basis

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{2}$$

- We define the Pauli matrices:

$$\sigma_x = X = |0\rangle\langle 1| + |1\rangle\langle 0| \tag{3}$$
$$\sigma_z = Z = |0\rangle\langle 0| - |1\rangle\langle 1| \tag{4}$$
$$\sigma_y = Y = iXZ \tag{5}$$

- We recall the following formula for any *unitary* vector $\vec{n}$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and I the identity matrix:

$$e^{i\alpha\vec{n}\cdot\vec{\sigma}} = \cos(\alpha)I + i\sin(\alpha)\vec{n}\cdot\vec{\sigma} \tag{6}$$

- Depending on the context, we use: $|\uparrow\rangle = |0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|\downarrow\rangle = |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$.

Problem 1. (10 points) *A dense-coding protocol with a third-party*

In this problem, we will revisit the **dense-coding protocol** between Alice and Bob but with an additional third-party: Charlie. The protocol works as follow:

1. Charlie is responsible for generating an entangled state:

$$|GHZ\rangle = \frac{1}{\sqrt{2}} \left(|000\rangle_{ABC} + |111\rangle_{ABC} \right)$$

   and distributes one qubit for Alice (A), one for Bob (B) and keeps one for himself (C).

2. Alice wants to send a message m of 2 classical bits. To this end, say $m = b_1 b_2$ where b_1 and b_2 are the two respective bits, she transforms her qubit with the operator $U = Z^{b_1} X^{b_2}$ and sends it to Bob.

3. Bob receives the qubit from Alice and make a measurement in the orthonormal basis $\{|\beta_{00}\rangle, |\beta_{10}\rangle, |\beta_{01}\rangle, |\beta_{11}\rangle\}$ given by:

$$|\beta_{00}\rangle = \left(\frac{|00\rangle_{AB} + |11\rangle_{AB}}{\sqrt{2}} \right), \qquad |\beta_{ij}\rangle = \left(Z^i X^j \otimes I \right) |\beta_{00}\rangle$$

(a) (1 point) What are the possible outcomes for Bob from his measurement?

(b) (3 points) For each $(i, j) \in \{(1, 0), (0, 1), (1, 1)\}$, express the value of $|\beta_{ij}\rangle$ in the computational basis.

(c) (3 points) Say Alice wants to send $m = 10$. What is the global state of the system after Alice's transformation and before Bob's measurement? Calculate the probability of the outcome $|\beta_{10}\rangle$ for Bob. Is he able to reconstruct the message m from Alice as seen in the dense coding protocol?

(d) (1 point) We will now see how Charlie can give a "key" to Bob in order for him to fully reconstruct the message of Alice. First of all, show that we have:

$$|GHZ\rangle = \frac{1}{\sqrt{2}} \left(|\beta_{00}\rangle \otimes |+\rangle + |\beta_{10}\rangle \otimes |-\rangle \right)$$

(e) (1 point) Assume now that Charlie makes a measurement on his qubit in the orthonormal basis $\{|+\rangle, |-\rangle\}$. Assume further that the outcome is $|+\rangle$. If Alice still wants to send $m = 10$, what is the probability of obtaining $|\beta_{10}\rangle$ for Bob?

(f) (1 point) Assume now that the qubit of Charlie collapsed to $|-\rangle$. What are the possible outcomes and their probabilities for Bob?

Solution to Problem 1:

(a) The measurements are the basis vectors $|\beta_{ij}\rangle$

(b)

$$|\beta_{10}\rangle = (Z_A \otimes I_B)|\beta_{00}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} \tag{7}$$

$$|\beta_{01}\rangle = (X_A \otimes I_B)|\beta_{00}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} \tag{8}$$

$$|\beta_{11}\rangle = (Z_A X_A \otimes I_B)|\beta_{00}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \tag{9}$$

(c) Alice applies Z_A to her qubit, so the global state of the system before Bob's measurement is:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$$

Bob's measurement are given by the projectors $P_{ij} = |\beta_{ij}\rangle\langle\beta_{ij}| \otimes I_C$ such that the outcome $|\beta_{10}\rangle$ has probability:

$$\mathcal{P}(|\psi_1\rangle \to |\beta_{10}\rangle) = \langle\psi_1| P_{10} |\psi_1\rangle \tag{10}$$

$$= \| P_{10} |\psi_1\rangle \|^2 \tag{11}$$

$$= \frac{1}{2} \| \langle\beta_{10}|00\rangle |\beta_{10}\rangle \otimes |0\rangle - \langle\beta_{10}|11\rangle |\beta_{10}\rangle \otimes |1\rangle \|^2 \tag{12}$$

$$= \frac{1}{2}\left(|\langle\beta_{10}|00\rangle|^2 + |\langle\beta_{10}|11\rangle|^2\right) \tag{13}$$

$$= \frac{1}{2} \tag{14}$$

So the answer is no, he cannot reconstruct the message m of Alice since there is at least one outcome with probability lower than 1.

(d) Direct calculation.

(e) In such case, the state vector before the application of the U for Alice is $|\psi_0\rangle = |\beta_{00}\rangle \otimes |+\rangle$. Thefore, we get $|\psi_1\rangle = (Z_A \otimes I_{BC})|\psi_0\rangle = |\beta_{10}\rangle \otimes |+\rangle$. Therefore, Bob will measure $|\beta_{10}\rangle$ with probability 1 and he has the message m of Alice through the convention that there is a direct correspondance between $b_1 b_2$ and the enumeration of the basis vector.

(f) Because Charlie has measured $|-\rangle$, the state vector becomes:

$$|\psi_1\rangle = (Z_A \otimes I_{BC})|\beta_{10}\rangle \otimes |-\rangle$$

So in fact:

$$|\psi_1\rangle = (Z_A \otimes I_{BC})((Z_A \otimes I)|\beta_{00}\rangle) \otimes |-\rangle = |\beta_{00}\rangle \otimes |-\rangle$$

Therefore, Bob will get the outcome $|\beta_{00}\rangle$ with probability 1. But now, upon knowing that Charlie measured $|-\rangle$, he can easily create a correspondence table for each case.

Problem 2. (10 points) *Spin dynamics: Ramsey sequence of operations*

Consider the Hamiltonian

$$H = \frac{\hbar\delta}{2}\sigma_z - \frac{\hbar\omega_1}{2}\sigma_x$$

Recall that in class we encountered this Hamiltonian as the one of a spin in a static along the z axis + rotating magnetic field in the (xy) plane. Here $\delta = \omega - \omega_0$ is the detuning parameter, between ω_0 the Larmor frequency and ω the frequency of the rotating field, whereas ω_1 is the strength of the rotating field.

But this Hamiltonian also models qubits or two energy levels of atoms in suitable regimes.

In this problem we consider the so-called *Ramsey sequence of operations*:

- A $\frac{\pi}{2}$ pulse: this is a time evolution during the time interval $[0, \tau]$ with $\tau = \frac{\pi}{2\omega_1}$ and $\delta = 0$.
- A Larmor precession during the time interval $[\tau, \tau + T]$ with $\omega_1 = 0$ and $\delta > 0$.
- A $\frac{\pi}{2}$ pulse as before during the time interval $[\tau + T, 2\tau + T]$.

We assume that the initial state of the spin is $|\uparrow\rangle$.

(a) (3 points) Compute the state at times τ, $\tau + T$, $2\tau + T$. *Hint*: we recall the formula for the time evolution operator $U_t = \exp(-i\frac{tH}{\hbar})$

(b) (3 points) Compute the probabilities that at the final time the spin is observed in states $|\uparrow\rangle$ or $|\downarrow\rangle$. Plot the probability $\mathbb{P}(|\uparrow\rangle_{t=0} \to |\downarrow\rangle_{t=2\tau+T})$ as function of T.

(c) (3 points) Illustrate the two trajectories of the spin on the Bloch spheres for $T = \frac{\pi}{\delta}$ and $T = \frac{2\pi}{\delta}$ and describe them in a few words as well.

(d) (1 point) Can you describe an analogy between the Ramsey sequence of operations and the Mach-Zehnder interferometer seen in class ?

Solution to problem 2:

(a) The evolution operator of the $\pi/2$ pulse is

$$U_{\pi/2} = \exp(i\frac{\tau\omega_1}{2}\sigma_x) = \begin{pmatrix} \cos\frac{\tau\omega_1}{2} & i\sin\frac{\tau\omega_1}{2} \\ i\sin\frac{\tau\omega_1}{2} & \cos\frac{\tau\omega_1}{2} \end{pmatrix}$$

For $\tau = \pi/2\omega_1$ we have

$$U_{\pi/2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

The evolution operator of the Larmor precession is

$$U_L = \exp(-i\frac{T\delta}{2}\sigma_z) = \begin{pmatrix} e^{-i\frac{T\delta}{2}} & 0 \\ 0 & e^{i\delta 2} \end{pmatrix} = e^{-i\frac{T\delta}{2}}\begin{pmatrix} 1 & 0 \\ 0 & e^{iT\delta} \end{pmatrix}$$

Thus the sequence of states is:

- after the first $\pi/2$ pulse

  $$U_{\pi/2}\left|\uparrow\right\rangle = \frac{1}{\sqrt{2}}(\left|\uparrow\right\rangle + i\left|\downarrow\right\rangle)$$

  This is a vector in the equator along the y axis of the Bloch sphere.

- at the end of the Larmor precession the vector has rotated to

  $$U_L\frac{1}{\sqrt{2}}(\left|\uparrow\right\rangle + i\left|\downarrow\right\rangle) = \frac{1}{\sqrt{2}}e^{-i\frac{T\delta}{2}}(\left|\uparrow\right\rangle + ie^{iT\delta}\left|\downarrow\right\rangle$$

  This is a vector in the equator making an angle $T\delta$ with y.

- After the last $\pi/2$ pulse

  $$U_{\pi/2}\frac{1}{\sqrt{2}}e^{-i\frac{T\delta}{2}}(\left|\uparrow\right\rangle + ie^{iT\delta}\left|\downarrow\right\rangle = e^{-i\frac{T\delta}{2}}(\frac{1-e^{iT\delta}}{2}\left|\uparrow\right\rangle + i\frac{1+e^{iT\delta}}{2}\left|\downarrow\right\rangle)$$

(b) By the Born rule the probabilities are

$$\mathbb{P}(\left|\uparrow\right\rangle_{t=0} \to \left|\uparrow\right\rangle_{t=2\tau+T}) = \left|\frac{1-e^{iT\delta}}{2}\right|^2 = \frac{1-\cos\delta}{2} = \sin^2\frac{T\delta}{2}$$

$$\mathbb{P}(\left|\uparrow\right\rangle_{t=0} \to \left|\downarrow\right\rangle_{t=2\tau+T}) = \left|\frac{1+e^{iT\delta}}{2}\right|^2 = \frac{1+\cos T\delta}{2} = \cos^2\frac{T\delta}{2}$$

The plot of the second probability is a periodic curve equal to (1 at $T = 0$), to (1/2 at $T = \pi/2\delta$), to (0 at $T = \pi/\delta$), to (1/2 at $T = 3\pi/2\delta$), to (1 at $T = 2\pi/\delta$). The period is $2\pi/\delta$.

(c) Here is a description of the two trajectories:

- For $T = \pi/\delta$ the full trajectory is first a rotation around x in the zy plane ending on the y axis, then a rotation of π around z ending up along -y, and finally a rotation around x in the yz plane ending in the initial $|\uparrow\rangle$ state along z.
- For $T = 2\pi/\delta$ the full trajectory is first a rotation around x in the zy plane ending on the y axis, then a full turn of 2π around z ending up along y, and finally a rotation around x in the yz plane ending in the down state $|\downarrow\rangle$ state along -z.

(d) The up and down spin states are the analog of the horizontal and vertical path states of the photon. The $\pi/2$ pulses are the analog of the two semi-transparent mirrors splitting the photon state along the two paths. The Larmor precession is the analog of the free travel (together with mirror reflection) of the photons.

Problem 3. (12 points) *Density matrix: a decoherence model*

In the following, we will study a model of decoherence of one qubit interacting with the environment. The whole system is defined in the hilbert space $\mathcal{H} = \mathcal{H}_\mathcal{E} \otimes \mathcal{H}_b$ where $\mathcal{H}_\mathcal{E}$ is the Hilbert space describing the possible states of the environment and $\mathcal{H}_b = \mathbb{C}^2$ is the Hilbert space describing the possible states of the qubit.

Let $|\phi_0\rangle = \alpha |0\rangle + \beta |1\rangle \in \mathcal{H}_b$ be the initial state of the qubit and $|\mathcal{E}\rangle \in \mathcal{H}_b$ that of the environment (or sometimes called *heat-bath*).

(a) (1 point) What is the initial global state $|\psi_0\rangle$ of the whole system?

Let $(|i\rangle)_{i\geq 1} \in \mathcal{H}_\mathcal{E}$ be an "infinite" orthonormal basis of the environment $\mathcal{H}_\mathcal{E}$. We define the evolution operator $U = \sum_{i=1}^{+\infty} |i\rangle \langle i| \otimes \mathcal{D}(\theta_i)$ for some distinct angles $\theta_i \in \mathbb{R}$, and the dephasing operator: $\mathcal{D}(\theta_i) = |0\rangle \langle 0| + e^{i\theta_i} |1\rangle \langle 1|$.

If the environment makes a transition from state $|\mathcal{E}\rangle$ to $|i\rangle$, we let $\mu(\theta_i) = P(|\mathcal{E}\rangle \to |i\rangle)$ the probability of such a transition.

(b) (2 points) Show that U is a unitary operator (describe your steps).

(c) (4 points) The state of the system evolves with a power $n \in \mathbb{N}$ of the operator U as $|\psi_n\rangle = U^n |\psi_0\rangle$. Show that $\mathcal{D}(\theta_i)^n = \mathcal{D}(n\theta_i)$ and deduce that

$$|\psi_n\rangle = \sum_{i=1}^{+\infty} e^{i \arg\langle i|\mathcal{E}\rangle} \sqrt{\mu(\theta_i)}\, |i\rangle \otimes (\mathcal{D}(n\theta_i) |\phi_0\rangle)$$

(d) (1 point) Now let's consider the density matrix of the qubit itself: $\rho_n = \mathrm{Tr}_{\mathcal{H}_\mathcal{E}} [|\psi_n\rangle \langle \psi_n|]$. First, using only the result of question (a), show that we have initially:

$$\rho_0 = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}$$

And give its Von Neumann entropy S_0.

(e) (1 point) For any angle $\theta \in \mathbb{R}$, show that we have:

$$\mathcal{D}(\theta)\rho_0\mathcal{D}(\theta)^\dagger = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* e^{-i\theta} \\ \alpha^*\beta e^{i\theta} & |\beta|^2 \end{pmatrix}$$

(f) (2 points) Now let's consider $\hat{\theta}$ a random variable in $\mathbb{R}$ with partial distribution function (PDF) $\theta \to \mu(\theta)$. Use the result of question (c) and (e) to show that the density matrix of the qubit coincide with the following expression:

$$\rho_n = \begin{pmatrix} |\alpha|^2 & \alpha\beta^*\mathbb{E}[e^{-in\hat{\theta}}] \\ \alpha^*\beta\mathbb{E}[e^{in\hat{\theta}}] & |\beta|^2 \end{pmatrix}$$

(g) (1 point) Now say that μ is the PDF of a gaussian distribution of mean 0 and variance σ^2. Show that the density matrix of the qubit evolves as:

$$\rho_n = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* e^{-\frac{1}{2}\sigma^2 n^2} \\ \alpha^*\beta e^{-\frac{1}{2}\sigma^2 n^2} & |\beta|^2 \end{pmatrix}$$

Calculate $\rho_\infty = \lim_{n\to\infty} \rho_n$ and give the associated entropy S_∞. Compare it with S_0 found in (d) and comment on the result.

Solution to Problem 3:

(a) The global state of the system is:

$$|\psi_0\rangle = |\mathcal{E}\rangle \otimes |\phi_0\rangle$$

(b) First of all:

$$U^\dagger = \left(\sum_{i=1}^{\infty} |i\rangle \langle i| \otimes \mathcal{D}(\theta_i)\right)^\dagger = \sum_{i=1}^{\infty} |i\rangle \langle i| \otimes \mathcal{D}(\theta_i)^\dagger$$

Thus because $(|i\rangle)$ is an orthonormal basis:

$$U^\dagger U = \sum_{i=1}^{\infty} |i\rangle \langle i| \otimes \mathcal{D}(\theta_i)^\dagger \mathcal{D}(\theta_i)$$

Now it is easy to show that $\mathcal{D}(\theta_i)^\dagger \mathcal{D}(\theta_i) = I$ so that $U^\dagger U = I$

(c) We have:

$$\mathcal{D}(\theta_i)^n = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta_i} \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 0 & e^{ni\theta_i} \end{pmatrix} = \mathcal{D}(n\theta_i)$$

Thus because $(|i\rangle)$ is an orthonormal basis:

$$U^n = \left(\sum_{i=1}^{\infty} |i\rangle \langle i| \otimes \mathcal{D}(\theta_i)\right)^n = \sum_{i=1}^{\infty} |i\rangle \langle i| \otimes (\mathcal{D}(\theta_i))^n = \sum_{i=1}^{\infty} |i\rangle \langle i| \otimes \mathcal{D}(n\theta_i)$$

Finally, because $\mathcal{P}(|\mathcal{E}\rangle \to |i\rangle) = |\langle i|\mathcal{E}\rangle|^2 = \mu(\theta_i)^2$ then $\langle i|\mathcal{E}\rangle = \sqrt{\mu(\theta_i)} e^{i \arg\langle i|\mathcal{E}\rangle}$ and thus we have:

$$|\psi_n\rangle = \sum_{i=1}^{\infty} |i\rangle \langle i|\mathcal{E}\rangle \otimes \mathcal{D}(n\theta_i) |\phi_0\rangle = \sum_{i=1}^{\infty} e^{i \arg\langle i|\mathcal{E}\rangle} \sqrt{\mu(\theta_i)}\, |i\rangle \otimes \mathcal{D}(n\theta_i) |\phi_0\rangle$$

(d) Using question (a) we find:

$$\rho_0 = |\phi_0\rangle \langle\phi_0| = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}$$

The Von Neumann entropy is $S_0 = 0$ as this is a rank-one matrix with one eigenvalue (=1).

(e) We find:

$$\mathcal{D}(\theta) |\phi_0\rangle = \begin{pmatrix} \alpha \\ e^{i\theta}\beta \end{pmatrix}$$

Thus:

$$\mathcal{D}(\theta)\rho_0\mathcal{D}(\theta)^\dagger = \begin{pmatrix} \alpha \\ e^{i\theta}\beta \end{pmatrix} \begin{pmatrix} \alpha^* & e^{-i\theta}\beta^* \end{pmatrix} = \begin{pmatrix} |\alpha|^2 & e^{-i\theta}\alpha\beta^* \\ e^{i\theta}\alpha^*\beta & |\beta|^2 \end{pmatrix}$$

(f) Using question (c) we have:

$$|\psi_n\rangle\langle\psi_n| = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} e^{i\arg\langle\mathcal{E}|\theta_i\rangle - i\arg\langle\mathcal{E}|\theta_j\rangle}\sqrt{\mu(\theta_i)\mu(\theta_j)}\,|j\rangle\langle i|\otimes\mathcal{D}(n\theta_j)|\phi_0\rangle\langle\phi_0|\mathcal{D}(n\theta_i)^\dagger$$

Therefore, using (e):

$$\rho_n = \sum_{i=1}^{\infty}\mu(\theta_i)\mathcal{D}(n\theta_i)\rho_0\mathcal{D}(n\theta_i)^\dagger \tag{15}$$

$$= \begin{pmatrix} \sum_{i=1}^{\infty}\mu(\theta_i)|\alpha|^2 & \sum_{i=1}^{\infty}\mu(\theta_i)\alpha\beta^* e^{-in\theta} \\ \sum_{i=1}^{\infty}\mu(\theta_i)\alpha\beta^* e^{in\theta} & \sum_{i=1}^{\infty}\mu(\theta_i)|\beta|^2 \end{pmatrix} \tag{16}$$

$$= \begin{pmatrix} \mathbb{E}_{\hat\theta}[|\alpha|^2] & \mathbb{E}_{\hat\theta}[\alpha\beta^* e^{-in\hat\theta}] \\ \mathbb{E}_{\hat\theta}[\alpha\beta^* e^{in\hat\theta}] & \mathbb{E}_{\hat\theta}[|\beta|^2] \end{pmatrix} \tag{17}$$

(g) This is a direct application of the MGF of $\hat\theta$. The limit is thus:

$$\rho_\infty = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix} \tag{18}$$

The entropy: $S_\infty = -|\alpha|^2\ln|\alpha|^2 - |\beta|^2\ln|\beta|^2 \geq 0 = S_0$

(h) We could in fact consider the R_y operator:

$$R_y(\theta) = \frac{1}{2}\begin{pmatrix} e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}} & -e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}} \\ e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}} & e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}} \end{pmatrix} \tag{19}$$

Thus we have:

$$R_y(\theta)|\phi_0\rangle = \frac{1}{2}\begin{pmatrix} \alpha(e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}}) + \beta(-e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}}) \\ \alpha(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}) + \beta(e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}}) \end{pmatrix} \tag{20}$$

$$= \frac{1}{2}\begin{pmatrix} (\alpha-\beta)e^{i\frac{\theta}{2}} + (\alpha+\beta)e^{-i\frac{\theta}{2}} \\ (\alpha+\beta)e^{i\frac{\theta}{2}} - (\alpha-\beta)e^{-i\frac{\theta}{2}} \end{pmatrix} \tag{21}$$

Then:

$$\langle\phi_0|R_y(\theta)^\dagger = \frac{1}{2}\begin{pmatrix} (\alpha^*-\beta^*)e^{-i\frac{\theta}{2}} + (\alpha^*+\beta^*)e^{i\frac{\theta}{2}} & (\alpha^*+\beta^*)e^{-i\frac{\theta}{2}} - (\alpha^*-\beta^*)e^{i\frac{\theta}{2}} \end{pmatrix} \tag{22}$$

In the limit $n\to\infty$, as we have seen, the terms for which we have an exponential vanishes, so:

$$\rho_\infty = \frac{1}{4}\begin{pmatrix} (\alpha-\beta)(\alpha^*-\beta^*) + (\alpha+\beta)(\alpha^*+\beta^*) & (\alpha-\beta)(\alpha^*+\beta^*) - (\alpha+\beta)(\alpha^*-\beta^*) \\ (\alpha+\beta)(\alpha^*-\beta^*) - (\alpha-\beta)(\alpha^*+\beta^*) & (\alpha-\beta)(\alpha^*-\beta^*) + (\alpha+\beta)(\alpha^*+\beta^*) \end{pmatrix} \tag{23}$$

So we find:

$$\rho_\infty = \begin{pmatrix} \frac{|\alpha|^2+|\beta|^2}{2} & i\Im(\alpha\beta^*) \\ -i\Im(\alpha\beta^*) & \frac{|\alpha|^2+|\beta|^2}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & i\Im(\alpha\beta^*) \\ -i\Im(\alpha\beta^*) & \frac{1}{2} \end{pmatrix} \tag{24}$$