ω vanishes along S (or vanishes at pts of S) if $\omega_p = 0$ for every $p \in S$. The weaker condition that $\iota^*\omega = 0$ is expressed by saying that the restriction of ω to S vanishes (or the pullback of ω to S vanishes).

Given a vector bundle $\pi_E : E \to M$, a subbundle of E is a vector bundle $\pi_D : D \to M$, in which D is a top. subspace of E and π_D is the restriction of π_E to D, such that for each $p \in M$, the subset $D_p = D \cap E_p$ is a linear subspace of E_p, and the vector space structure on D_p is the one inherited from E_p. Note that the condition that D be a vector bundle over M implies that all of the fibers D_p are non-empty and have the same dimension.

If $E \to M$ is a smooth vector bundle, then a subbundle of E is called a smooth subbundle if it is a smooth vector bundle and an embedded submfd of E.

The following lemma [Lee, Lemma 10.32] gives a convenient condition for checking that a union of subspaces $\{D_p \subseteq E_p \mid p \in M\}$ is a smooth subbundle.

· LEM (Local frame criterion for subbundles): Let $\pi : E \to M$ be a smooth vector bundle. Suppose that for each $p \in M$ we are given an m-dim. linear subspace $D_p \subseteq E_p$. Then $D = \bigcup_{p \in M} D_p \subseteq E$ is a smooth subbundle of E iff the following condition is satisfied:

"Each pt of M has a neighborhood U on which there exist smooth local sections $\sigma_1, \dots, \sigma_m : U \longrightarrow E$ with the property that $\sigma_1(q), \dots, \sigma_m(q)$ form a basis for D_q at each $q \in U$."

DEF. 8.15: Let M be a smooth mnfd.

(a) We define the bundle of covariant k-tensors on M by

$$T^k(T^*M) := \coprod_{p \in M} T^k(T_p^*M)$$

with the obvious projection map, which is often also called a tensor bundle over M. Its sections are called (covariant k-) tensor fields on M.

(b) The subset of $T^k(T^*M)$ consisting of alternating k-tensors is denoted by $\Lambda^k(T^*M)$:

$$\Lambda^k(T^*M) := \coprod_{p \in M} \Lambda^k(T_p^*M).$$

It can be shown (exercise!) that $\Lambda^k(T^*M)$ is a smooth subbundle of $T^k(T^*M)$, and thus it is a smooth vector bundle of rk $\binom{n}{k}$ over M. Its sections are called (differential) k-forms on M; they are (cont.) tensor fields whose value at each pt is an alternating k-tensor. The integer k is called the degree of the form. We denote the vector space of smooth k-forms by

$$\underline{\Omega}^k(M) = \Gamma\left(\Lambda^k(T^*M)\right).$$

Note that a 0-form is just a cont real-valued fnct on M (because $\Lambda^0(T^*M) = \coprod_{p\in M} \Lambda^0(T_p^*M) = \coprod_{p\in M} \mathbb{R} = M \times \mathbb{R}$), see ES10E3(c), and a 1-form is a covector field on M (since $\Lambda^1(T^*M) = \coprod_{p\in M} \Lambda^1(T_p^*M) \cong \coprod_{p\in M} T_p^*M = T^*M$).

The <u>wedge product</u> of two differential forms is defined pointwise: $(\omega \wedge \eta)_p := \omega_p \wedge \eta_p$. Thus, the wedge product of a k-form with an ℓ-form is a $(k+\ell)$-form. If f is a 0-form and η is a k-form, then we interpret the wedge product $f \wedge \eta$ to mean the ordinary product $f\eta$; see lecture 10, p. 85. If we define

$$\underline{\Omega}^*(M) := \bigoplus_{k=0}^{n} \underline{\Omega}^k(M),$$

then the wedge product turns $\underline{\Omega}^*(M)$ into an associative, anticommutative, graded $\mathbb{R}$-algebra.

In any smooth chart $(U, (x^i))$, a k-form ω can be written as

$$\omega = \sum_I{}' \omega_I \, dx^{i_1} \wedge \ldots \wedge dx^{i_k} = \sum_I{}' \omega_I \, dx^I,$$

where the coefficients ω_I are smooth fncts defined on the coordinate domain U, and we use dx^I as an abbreviation for $dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ (where $I = (i_1, \ldots, i_k)$), and the primed summation sign denotes a sum over only increasing multi-indices. According to PROP. 6.13, ω is smooth iff the component fncts ω_I are smooth. Since

$$dx^{i_1} \wedge \ldots \wedge dx^{i_k}\left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}}\right) = \delta^I_J,$$

see [Multilinear algebra, Lemma 19(c)], the component fncts ω_I of ω are determined by

$$\omega_I = \omega\left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}}\right).$$

If $F: M \to N$ is a smooth map and ω is a differential form on N, then $F^*\omega$ is a differential form on M, defined as follows:

$$(F^*\omega)_p(v_1, \dots, v_k) := \omega_{F(p)}\left(dF_p(v_1), \dots, dF_p(v_k)\right).$$

LEM 8.16: Let $F: M \to N$ be a smooth map. The following statements hold:

(a) $F^*: \underline{\Omega}^k(N) \to \underline{\Omega}^k(M)$ is linear over $\mathbb{R}$.

(b) $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$

(c) In any smooth chart $(V, (y^i))$ for N, we have

$$F^*\left(\sum_I{}' \omega_I \, dy^{i_1} \wedge \dots \wedge dy^{i_k}\right) = \sum_I{}' (\omega_I \circ F)\, d(y^{i_1} \circ F) \wedge \dots \wedge d(y^{i_k} \circ F).$$

PROOF: ES13-II-E1. ■

This lemma gives a computational rule for pullbacks of differential forms similar to the one we developed earlier for covector fields.

EXAMPLE 8.17: Consider the smooth fnct

$$F: \mathbb{R}^2 \to \mathbb{R}^3, \quad (u,v) = (u, v, u^2 - v^2)$$

and the smooth 2-form

$$\omega = y\, dx \wedge dz + x\, dy \wedge dz \in \underline{\Omega}^2(\mathbb{R}^3).$$

Then

$$F^*\omega = F^*(y\,dx\wedge dz + x\,dy\wedge dz)$$
$$= v\,du\wedge d(u^2-v^2) + u\,dv\wedge d(u^2-v^2)$$
$$= v\,du\wedge(2u\,du - 2v\,dv) + u\,dv\wedge(2u\,du-2v\,dv)$$
$$\underset{dv\wedge dv=0}{\overset{du\wedge du=0}{=\!=\!=}} -2v^2\,du\wedge dv + 2u^2\,dv\wedge du$$
$$\underset{-dv\wedge du}{\overset{du\wedge dv=}{=\!=\!=}} -2(u^2+v^2)\,du\wedge dv.$$

$\rightsquigarrow$ see also ES13-II-E2 for an example regarding the change of coordinates

· <u>PROP. 8.18</u> (Pullback formula for top degree forms): Let $F: M \to N$ be a smooth map between smooth n-mnfds. If (x^i) and (y^j) are smooth coordinates on open subsets $U \subseteq M$ and $V \subseteq N$, respectively, and u is a cont. real-valued fnct on V, then the following holds on $U \cap F^{-1}(V)$:

$$F^*(u\,dy^1\wedge\ldots\wedge dy^n) = (u\circ F)\det DF\,(dx^1\wedge\ldots\wedge dx^n), \qquad (*_7)$$

where DF represents the Jacobian matrix of F in these coordinates.

<u>PROOF</u>: Since the fiber of $\Lambda^n(T^*M)$ is spanned by $dx^1\wedge\ldots\wedge dx^n$ at each pt, it suffices t.s.t. both sides of $(*_7)$ agree when evaluated on $(\partial/\partial x^1, \ldots, \partial/\partial x^n)$. We have

$$F^*(u\,dy^1\wedge\ldots\wedge dy^n) \underset{8.16(c)}{\overset{LEM}{=\!=\!=}} (u\circ F)\,d(\underbrace{y^1\circ F}_{F^1})\wedge\ldots\wedge d(\underbrace{y^n\circ F}_{F^n}) \Rightarrow$$

$$\xrightarrow[\text{Prop. 24(c)(d)]}]{\text{[Mult. alg.,}} F^*(u\,dy^1\wedge\dots\wedge dy^n)\left(\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^n}\right) =$$

$$= (u\circ F)\; dF^1\wedge\dots\wedge dF^n\left(\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^n}\right)$$

$$= (u\circ F)\; \det\left(dF^j\left(\frac{\partial}{\partial x^i}\right)\right)$$

$$= (u\circ F)\; \det\left(\frac{\partial F^j}{\partial x^i}\right)\cdot 1 \qquad 1 = dx^1\wedge\dots\wedge dx^n\left(\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^n}\right)$$

$$= (u\circ F)\; \det DF\; (dx^1\wedge\dots\wedge dx^n)\left(\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^n}\right),$$

as desired. ∎

· **COR. 8.19**: If $(U,(x^i))$ and $(\tilde U,(\tilde x^j))$ are overlapping smooth coordinate charts on M, then the following identity holds on $U\cap\tilde U$:

$$d\tilde x^1\wedge\dots\wedge d\tilde x^n = \det\left(\frac{\partial\tilde x^j}{\partial x^i}\right) dx^1\wedge\dots\wedge dx^n.$$

PROOF: Apply PROP. 8.18 for $F = Id_{U\cap\tilde U}$, but using coordinates (x^i) in the domain and $(\tilde x^j)$ in the codomain. ∎

We now define a natural differential operator on smooth forms, called the exterior derivative, which is a generalization of the differential of a fnct. More precisely, for each smooth mnfd M, we will show that there is a differential operator $d:\Omega^k(M)\to\Omega^{k+1}(M)$ satisfying $d(d\omega)=0$ for all ω.

The dfn of d on Euclidean space is straightforward: if $\omega = \sum_J{}'\,\omega_J\,dx^J$ is a smooth k-form on an open subset $U\subseteq\mathbb{R}^n$, we define its <u>exterior derivative</u> $d\omega$ to be the following

$(k+1)$-form:

$$d\left(\sum_J{}' \omega_J dx^J\right) = \sum_J{}' d\omega_J \wedge dx^J, \qquad (*_8)$$

where $d\omega_J$ is the differential of the smooth fnct ω_J. In somewhat more detail, this is

$$d\left(\sum_J{}' \omega_J dx^{j_1} \wedge \ldots \wedge dx^{j_k}\right) =$$

$$= \sum_J \sum_i \frac{\partial \omega_J}{\partial x^i} dx^i \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_k}.$$

For instance, for a smooth 0-form f we have

$$df = \frac{\partial f}{\partial x^i} dx^i,$$

which is just the differential of f (see $(*_4)$ on p. 112), and for a smooth 1-form ω we compute that

$$d\omega = \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j}\right) dx^i \wedge dx^j.$$

In order to transfer this definition to mnfds, we first need to check that it satisfies the following properties.

· <u>PROP. 8.20</u> (Properties of the exterior derivative on $\mathbb{R}^n$)

(a) d is $\mathbb{R}$-linear.

(b) If ω is a smooth k-form and η is a smooth ℓ-form on an open subset $U \subseteq \mathbb{R}^n$, then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(c) $d \circ d \equiv 0$.

(d) d commutes with pullbacks: if $F: U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ is a smooth map, and $\omega \in \underline{\Omega}^k(V)$, then

$$F^*(d\omega) = d(F^*\omega).$$

<u>PROOF</u>:

(a) Follows immediately from the dfn.

(b) Due to (a), it suffices to consider terms of the form $\omega = u\,dx^I \in \underline{\Omega}^k(U)$ and $\eta = v\,dx^J \in \underline{\Omega}^\ell(U)$, where $u, v \in C^\infty(U)$.

- Claim: For any multi-index I we have

$$d(u\,dx^I) = du \wedge dx^I.$$

- Proof: If I has repeated indices, then clearly $d(u\,dx^I) = 0 = du \wedge dx^I$. Otherwise, let σ be a permutation sending I to an increasing multi-index J. Then

$$\begin{aligned} d(u\,dx^I) &= \operatorname{sgn}(\sigma)\, d(u\,dx^J) = \operatorname{sgn}(\sigma)\, du \wedge dx^J \\ &= du \wedge dx^I. \end{aligned}$$

•

Using the claim, we compute

$$\begin{aligned} d(\omega \wedge \eta) &= d\big((u\,dx^I) \wedge (v\,dx^J)\big) \\ &= d(uv\,dx^I \wedge dx^J) \overset{\text{dfn}}{=} \\ &= (v\,du + u\,dv) \wedge dx^I \wedge dx^J \qquad \frac{du \wedge dx^I =}{= (-1)^k dx^I \wedge du} \\ &= (du \wedge dx^I) \wedge (v\,dx^J) + (-1)^k (u\,dx^I) \wedge (dv \wedge dx^J) \\ &\overset{\text{claim}}{=} \underbrace{d(u\,dx^I)}_{d\omega} \wedge \underbrace{(v\,dx^J)}_{\eta} + (-1)^k \underbrace{(u\,dx^I)}_{\omega} \wedge \underbrace{d(v\,dx^J)}_{d\eta} \end{aligned}$$

(c) We first deal with the case of a smooth 0-form u:

$$d(du) = d\left(\frac{\partial u}{\partial x^i} dx^i\right) = \frac{\partial^2 u}{\partial x^i \partial x^j} dx^i \wedge dx^j \overset{dx^i \wedge dx^i = 0}{=\!=\!=}$$

$$= \sum_{i<j} \left(\frac{\partial^2 u}{\partial x^i \partial x^j} - \frac{\partial^2 u}{\partial x^j \partial x^i}\right) dx^i \wedge dx^j = 0.$$

We now deal with the general case $(u \in \Omega^k(U))$:

$$d(d\omega) = d\left(\sum_J{}' d\omega_J \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_k}\right)$$

$$\overset{(a)}{\underset{(b)}{=\!=}} \sum_J{}' \underbrace{d(d\omega_J)}_{\to 0 \text{ by case } k=0} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_k} +$$

$$+ \sum_J{}' (-1) \cdot d\omega_J \wedge \underbrace{d(dx^{j_1} \wedge \ldots \wedge dx^{j_k})}_{\to 0 \text{ by (b) and by case } k=0}$$

$$= 0.$$

(d) Due to (a), it suffices to consider $\omega = u\, dx^{i_1} \wedge \ldots \wedge dx^{i_k}$, in which case we have

$$F^*\left(d(u\,dx^{i_1} \wedge \ldots \wedge dx^{i_k})\right) = F^*\left(du \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}\right) \overset{\text{LEM 8.16(b)(c)}}{\underset{\text{PROP. 8.11}}{=\!=\!=}}$$

$$= d(u \circ F) \wedge d(x^{i_1} \circ F) \wedge \ldots \wedge d(x^{i_k} \circ F) \overset{(*)}{=\!=}$$

$$= d\left((u \circ F)\, d(x^{i_1} \circ F) \wedge \ldots \wedge d(x^{i_k} \circ F)\right) \overset{\text{LEM}}{\underset{8.16(c)}{=\!=\!=}}$$

$$= d\left(F^*(u\,dx^{i_1} \wedge \ldots \wedge dx^{i_k})\right),$$

so we are done. ∎

$(*)$: We have an expression of the form $df \wedge \eta$, where $\eta = dg_1 \wedge \ldots \wedge dg_k$, so $d(f\eta) \overset{P.119}{=\!=} d(f \wedge \eta) \overset{(b)}{=} df \wedge \eta + (-1)^0 f \wedge d\eta$ $= df \wedge \eta$, since $d\eta = 0$ by (b) and (c).

Here, $f := u \circ F$ and $\eta := d(x^{i_1} \circ F) \wedge \ldots \wedge d(x^{i_k} \circ F)$.

THM 8.21 (Existence and uniqueness of exterior differentiation):
Let M be a smooth mnfd. For each k there are unique operators

$$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M),$$

called <u>exterior differentiation</u>, satisfying the following properties:

(a) d is $\mathbb{R}$-linear.

(b) If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta .$$

(c) $d \circ d \equiv 0$.

(d) For $f \in \Omega^0(M) = C^\infty(M)$, df is the differential of f, given by $df(X) = Xf$.

In any smooth chart, d is given by $(*_8)$.

PROOF:

- Existence: Given $\omega \in \Omega^k(M)$, for each smooth chart (U,φ) for M, we set $d\omega := \varphi^* d((\varphi^{-1})^* \omega)$. This is well-defined, since for any other smooth chart (V,ψ) the map $\varphi \circ \psi^{-1}$ is a diffeomorphism between open subsets of $\mathbb{R}^n$, so PROP. 8.20(d) yields

$$\begin{aligned}
\psi^* d((\psi^{-1})^*\omega) &= (\varphi^{-1} \circ \varphi)^* \psi^* d((\psi^{-1})^*\omega) \\
&= \varphi^* \circ (\varphi^{-1})^* \circ \psi^* d((\psi^{-1})^*\omega) \qquad \underline{(\varphi^{-1})^* \circ \psi^* =} \\
&\hphantom{=} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad = (\psi \circ \varphi^{-1})^* \\
&= \varphi^* d\big(\underbrace{(\psi \circ \varphi^{-1})^* (\psi^{-1})^*}_{(\psi^{-1} \circ \psi \circ \varphi^{-1})^*} \omega\big) \\
&= \varphi^* d((\varphi^{-1})^*\omega).
\end{aligned}$$

Moreover, d satisfies (a)-(d) by virtue of PROP. 8.20.

- Uniqueness: Suppose that d is any operator satisfying (a)-(d). We first show that d is determined locally: if ω_1 and ω_2 are k-forms that agree on an open subset $U \subseteq M$, then $d\omega_1 = d\omega_2$ on U. Let $p \in U$, set $\eta := \omega_1 - \omega_2$ and let $\psi \in C^\infty(M)$ be a bump fnct that is identically 1 on some neighborhood of p and supported in U. Then $\psi\eta$ is identically zero, so (a)-(d) imply that $0 = d(\psi\eta) = d\psi \wedge \eta + \psi d\eta$. Evaluating this at p and using that $\psi(p) = 1$ and $d\psi_p = 0$, we conclude that $0 = d\eta_p = d\omega_1|_p - d\omega_2|_p$.

Now let $\omega \in \Omega^k(M)$ and let (U, φ) be a smooth chart on M. We write ω in coordinates as $\sum_I' \omega_I dx^I$. For any $p \in U$ by means of a bump fnct we construct global smooth fncts $\tilde{\omega}_I$ and $\tilde{x}^i$ on M that agree with ω_I and dx^i in a neighborhood of p. By virtue of (a)-(d) together with the observation in the previous paragraph, it follows that $(*_8)$ holds at p. Since p was arbitrary, this d must be equal to the one we defined above. ■

⇝ The differential on fncts extends uniquely to an anti-derivation of $\Omega^*(M)$ of degree $+1$ whose square is zero.

· **PROP. 8.22** (Naturality of the exterior derivative): If $F: M \to N$ is a smooth map, then for each k the pullback map $F^*: \Omega^k(N) \to \Omega^k(M)$ commutes with d, i.e.,

$$F^*(d\omega) = d(F^*\omega), \ \forall \omega \in \Omega^k(N).$$

PROOF: We apply PROP. 8.20(d) to the coordinate representation $\psi \circ F \circ \varphi^{-1}$ of F and on $U \cap F^{-1}(V)$ we obtain

$$\begin{aligned} F^*(d\omega) &= F^* \psi^* d((\psi^{-1})^*\omega) \\ &= \varphi^* \circ (\psi \circ F \circ \varphi^{-1})^* d((\psi^{-1})^*\omega) \\ &= \varphi^* d\left((\psi \circ F \circ \varphi^{-1})^* (\psi^{-1})^*\omega\right) \\ &= \varphi^* d((\varphi^{-1})^* F^*\omega) \\ &= d(F^*\omega). \end{aligned}$$

⇝ compute exterior derivatives of k-forms on $\mathbb{R}^3$

· **DEF. 8.23**: Let M be a smooth mnfd and let $\omega \in \Omega^k(M)$. We say that ω is closed if $d\omega = 0$, and exact if there exists $\eta \in \Omega^{k-1}(M)$ s.t. $\omega = d\eta$.

· **REM. 8.24**: Every exact form is closed, since $d \circ d \equiv 0$, but the converse need not be true in general. However, it can be shown that closed forms are locally exact (but not necessarily globally).