PROOF: (Similar to the proof of PROP. 6.5) Given a smooth chart (U, φ) for M, with coordinate fncts (x^i), define

$$\Phi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^n$$
$$\xi_i \lambda^i|_p \longmapsto (p, (\xi_1, \dots, \xi_n)),$$

where λ^i is the i-th coordinate covector field associated with (x^i). Suppose that $(\tilde{U}, \tilde{\varphi})$ is another smooth chart for M with coordinate fncts $(\tilde{x}^j)$, and let $\tilde{\Phi} : \pi^{-1}(\tilde{U}) \longrightarrow \tilde{U} \times \mathbb{R}^n$ be defined analogously. On $\pi^{-1}(U \cap \tilde{U})$, it follows from $(*_2)$ that

$$\left(\Phi \circ \tilde{\Phi}^{-1}\right)\left(p, (\tilde{\xi}^1, \dots, \tilde{\xi}^n)\right) = \left(p, \left(\frac{\partial \tilde{x}^j}{\partial x^1}(p)\tilde{\xi}_j, \dots, \frac{\partial \tilde{x}^j}{\partial x^n}(p)\tilde{\xi}_j\right)\right).$$

The $GL(n, \mathbb{R})$-valued fnct $(\partial \tilde{x}^j / \partial x^i)$ is smooth, so it follows from the vector bundle chart lemma (= LEM 6.7) that T^*M has a smooth structure making it into a smooth vector bundle for which the maps Φ are local trivializations. Uniqueness follows as in the proof of ES11E1. ∎

L12

L13

As in the case of the tangent bundle, smooth local coordinates for M yield smooth local coordinates for its cotangent bundle. If (x^i) are smooth coordinates on an open subset $U \subseteq M$, ES10ES(d) shows that the map

$$\pi^{-1}(U) \longrightarrow \mathbb{R}^{2n}, \quad \xi_i \lambda^i|_p \longmapsto \left(x^1(p), \dots, x^n(p), \xi_1, \dots, \xi_n\right)$$

is a smooth coordinate chart for T^*M. We call (x^i, ξ_i) <u>the natural coordinates for T^*M associated with (x^i)</u>.

· <u>DEF. 8.4</u>: A rough / cont. / smooth (local or global) section of T^*M is called a <u>rough / cont. / smooth covector field</u> or a <u>(differential) 1-form</u> on a smooth mnfd M.

The set $\mathfrak{X}^*(M)$ of all smooth (global) covector fields on a smooth mnfd M is an infinite-dimensional $\mathbb{R}$-v.s. and a module over the ring $C^\infty(M)$. (ES10E3)

⇝ <u>local / global coframe for M</u> = local / global frame for T^*M; see DEF. 6.11

⇝ completion of smooth local coframes for M: special case of ES10E4

· <u>EXAMPLE 8.5</u>: For any smooth chart $(U,(x^i))$, the coordinate covector fields (λ^i) defined above constitute a local coframe over U, called a <u>coordinate coframe</u>. By ES13-I-E1 = PROP. 8.6, every coordinate coframe is smooth, because its component fncts in the given chart are constants.

In any smooth local coordinates (x^i) on an open subset $U \subseteq M$, a (rough) covector field ω can be written in terms of the coordinate covector fields (λ^i) as $\omega = \omega_i \lambda^i$ for n fncts $\omega_i : U \to \mathbb{R}$, called <u>the component fncts of ω in the given chart</u> and characterized by

$$\omega_i(p) = \omega_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right).$$

If ω is a (rough) covector field and if X is a (rough) vector field on M, then we can form a fnct

$$\omega(X) : M \longrightarrow \mathbb{R}, \quad p \mapsto \omega_p(X_p).$$

If we write $\omega = \omega_i \lambda^i$ and $X = X^j \frac{\partial}{\partial x^j}$ in terms of local coordinates, then $\omega(X)$ has the local coordinate representation

$$\omega(X) = \omega_i X^i.$$

Just as in the case of vector fields (see PROP. 7.3 and 7.5), there are several ways to check smoothness of covector fields.

· PROP. 8.6 (Smoothness criteria for covector fields): Let M be a smooth mnfd and let $\omega : M \longrightarrow T^*M$ be a rough covector field. T.f.a.e.

(a) ω is smooth.

(b) In every smooth chart, the component fncts of ω are smooth.

(c) Each pt of M is contained in some coordinate chart in which ω has smooth component fncts.

(d) For every $X \in \mathfrak{X}(M)$, the fnct $\omega(X) : M \longrightarrow \mathbb{R}$ is smooth.

(e) For every open subset $U \subseteq M$ and every smooth vector field X on U, the fnct $\omega(X) : U \longrightarrow \mathbb{R}$ is smooth.

PROOF: ES13 - I - E1. ∎

Since any open subset of a smooth mnfd is again a smooth mnfd, PROP. 8.6 applies equally well to covector fields defined only on some open subset of M.

$\rightarrow \exists$ local coframe criterion for cont / smoothness of rough covector fields : special case of PROP. 6.13.

$$\left(\begin{array}{l}\text{frame } (E_i) \text{ for } TM \rightsquigarrow \text{coframe } (\varepsilon^i) \text{ dual to } (E_i) \\ \text{coframe } (\varepsilon^i) \text{ for } T^*M \rightsquigarrow \text{frame } (E_i) \text{ dual to } (\varepsilon^i)\end{array}\right)$$

The most important application of covector fields is that they enable us to interpret in a coordinate-independent way the partial derivatives of a smooth fnct as the components of a covector field.

Let $f \in C^\infty(M)$. We define a covector field df, called <u>the differential of f at $p \in M$</u>, by

$$df_p(v) = vf \ , \ v \in T_pM.$$

<u>PROP. 8.7</u> : The differential of a smooth fnct is a smooth covector field.

<u>PROOF</u> : It is straightforward to check that $df_p \in T_p^*M$ for all $p \in M$. To verify that df is smooth we apply PROP. 8.6(d) : for any $X \in \mathfrak{X}(M)$, the fnct $df(X)$ is smooth, because it is equal to Xf (see PROP. 7.5). ∎

For a smooth real-valued fnct $f : M \longrightarrow \mathbb{R}$, we now have two different definitions for the differential of f at $p \in M$. In Ch. 3 we defined df_p as a linear map $T_pM \longrightarrow T_{f(p)}\mathbb{R}$, while here we defined df_p as a covector at $p \in M$, i.e., a linear map $T_pM \longrightarrow \mathbb{R}$. These are really the same object, once we take into account the identification between $T_{f(p)}\mathbb{R}$ and $\mathbb{R}$; one easy way to see this is to note that both are represented in coordinates by the row matrix whose components are the partial derivatives of f. (Let us verify this below for df defined as above.) ↓

Let us compute the coordinate representation of df. Let (x^i) be smooth coordinates on an open subset $U \subseteq M$ and let (λ^i) be the corresponding coordinate coframe on U. Write df in coordinates as $df_p = A_i(p)\,\lambda^i|_p$ for some fncts $A_i : U \longrightarrow \mathbb{R}$. Then the dfn of df implies

$$A_i(p) = df_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \frac{\partial}{\partial x^i}\bigg|_p f = \frac{\partial f}{\partial x^i}(p),$$

which yields the following formula for the coordinate representation of df:

$$df_p = \frac{\partial f}{\partial x^i}(p)\,\lambda^i\big|_p \qquad (*_3)$$

Thus, the component fncts of df in any smooth coordinate chart are the partial derivatives of f w.r.t. those coordinates. Due to this, we can think of df as an analogue of the classical gradient (the vector field in $\mathbb{R}^n$ whose components are the partial derivatives of the fnct), reinterpreted in a way that makes coordinate-independent sense on a mnfd.

If we apply $(*_3)$ to the special case in which f is one of the coordinate fncts $x^j : U \longrightarrow \mathbb{R}$, we obtain

$$dx^j\big|_p = \frac{\partial x^j}{\partial x^i}(p)\,\lambda^i\big|_p = \delta^j_i\,\lambda^i\big|_p = \lambda^j\big|_p\,;$$

in other words, the coordinate vector field λ^j is none other than the differential dx^j. Therefore, $(*_3)$ can be rewritten as

$$df_p = \frac{\partial f}{\partial x^i}(p)\, dx^i|_p \quad , p \in U,$$

or as an equation between covector fields instead of covectors:

$$df = \frac{\partial f}{\partial x^i} dx^i . \qquad (*_4)$$

In particular, in the 1-dim case, this reduces to

$$df = \frac{df}{dx} dx .$$

Thus, we have recovered the familiar classical expression for the differential of a fnct f in coordinates. Henceforth, we abandon the notation λ^i for the coordinate coframe, and use dx^i instead.

· EXAMPLE 8.8: If

$$f : \mathbb{R}^2 \to \mathbb{R} , \ (x,y) \mapsto x^2 y \cos x ,$$

then

$$df = \frac{\partial(x^2 y \cos x)}{\partial x} dx + \frac{\partial (x^2 y \cos x)}{\partial y} dy$$
$$= \left(2xy\cos x - x^2 y \sin x\right) dx + \left(x^2 \cos x\right) dy .$$

⇝ properties of the differential = ES13 - I - E2

⇝ derivative of a fnct along a curve = ES13 - I - E3(a)

DEF. 8.9: Let $F: M \to N$ be a smooth map and let $p \in M$. The differential $dF_p: T_pM \to T_{F(p)}N$ yields a dual linear map, $dF_p^*: T^*_{F(p)}N \to T^*_pM$, called <u>the (pointwise) pullback by F at p</u> (or the cotangent map of F) and characterized by

$$dF_p^*(\omega)(v) = \omega(dF_p(v)), \omega \in T^*_{F(p)}N, v \in T_pM.$$

Unlike vector fields, whose pushforwards are defined only in special cases (see, e.g., ES11E4), covector fields always pullback to covector fields.

DEF. 8.10: Let $F: M \to N$ be a smooth map and let $\omega: N \to T^*N$ be a rough covector field. We define a rough covector field $F^*\omega$ on M, called <u>the pullback of ω by F</u>, by

$$(F^*\omega)_p := dF_p^*(\omega_{F(p)}). \qquad (*_5)$$

It acts on a vector $v \in T_pM$ by

$$(F^*\omega)_p(v) = \omega_{F(p)}(dF_p(v)).$$

PROP. 8.11: Let $F: M \to N$ be a smooth map and let ω be a covector field on N. If $u: N \to \mathbb{R}$ is a cont. fnct, then

$$F^*(u\omega) = (u \circ F) F^*\omega.$$

If additionally u is smooth, then

$$F^*(du) = d(u \circ F).$$

PROOF: We have

$$F^*(u\omega)_p \overset{(*_5)}{=} dF_p^*((u\omega)_{F(p)}) = dF_p^*(u(F(p))\,\omega_{F(p)}) \overset{\text{lin}}{=}$$

$$= u(F(p))\, dF_p^*(\omega_{F(p)}) \overset{(*_5)}{=} (u \circ F)(p)\,(F^*\omega)_p$$
$$= \big((u\circ F)(F^*\omega)\big)_p ,$$

which proves the first statement. Now, for the second statement, if $p \in M$ and $v \in T_pM$, then

$$(F^* du)_p(v) \overset{(*_5)}{=} \big(dF_p^*(du_{F(p)})\big)(v) \overset{\text{dfn}}{=}$$
$$= du_{F(p)}(dF_p(v)) \quad \overset{\text{dfn of } du}{=}$$
$$= dF_p(v)\cdot u \quad \overset{\text{dfn of } dF_p}{=}$$
$$= v(u \circ F) \quad \overset{\text{dfn of } d(u\circ F)}{=}$$
$$= d(u\circ F)_p(v),$$

immediately using ESHEL(b) and the identit.

which yields the second statement. ∎

· PROP. 8.19: Let $F: M \to N$ be a smooth map and let ω be a (cont.) covector field on N. Then $F^*\omega$ is a (cont.) covector field on M, and if ω is smooth, then so is $F^*\omega$.

PROOF: Fix $p \in M$ and choose smooth coordinates (y^j) for N in a neighborhood V of $F(p)$. Set $U := F^{-1}(V)$ and observe that U is a neighborhood of p in M. Writing ω in coordinates as $\omega = \omega_j dy^j$ for (cont.) fncts on V and using PROP. 8.11 twice (for $F|_U$), we compute that

$$F^*\omega = F^*(\omega_j dy^j) = (\omega_j \circ F)\, F^*dy^j = (\omega_j \circ F)\, d(y^j \circ F) \qquad (*_6)$$

This expression is continuous, and it is smooth when ω is smooth, so we are done. ∎

Formula $(\star_6)$ for the pullback of a covector field can also be written in the following way:

$$F^*\omega = (\omega_j \circ F)\, d(y^j \circ F) = (\omega_j \circ F)\, dF^j ,$$

where F^j is the j-th component fnct of F in these coordinates. Using either of these formulas, the computation of pullbacks in coordinates is quite simple.

· EXAMPLE 8.13: Consider the smooth map

$$F : \mathbb{R}^3 \longrightarrow \mathbb{R}^2, \quad (x,y,z) \longmapsto (x^2y, y\sin z) = (u,v)$$

and the smooth covector field

$$\omega = u\,dv + v\,du \in \mathfrak{X}^*(\mathbb{R}^2).$$

According to $(\star_6)$, we have

$$\begin{aligned} F^*\omega &= (u \circ F)\, d(v \circ F) + (v \circ F)\, d(u \circ F) \\ &= (x^2y)\, d(y\sin z) + (y\sin z)\, d(x^2y) \\ &= (x^2y)(\sin z\, dy + y\cos z\, dz) + y\sin z\,(2xy\,dx + x^2 dy) \\ &= (2xy^2\sin z)\,dx + (2x^2y\sin z)\,dy + (x^2y^2\cos z)\,dz . \end{aligned}$$

In other words, to compute $F^*\omega$, all we need to do is substitute the component fncts of F for the coordinate fncts of N everywhere they appear in ω.

⇝ see also [Lee, Example 11.28] for an example about the transformation law for a covector field under a change of coordinates

In Ch. 7 (see also ES12E1) we considered the conditions under which a vector field restricts to a submnfd. The restriction of covector fields to submnfds is much simpler and will be briefly discussed below (see also ES13-I-E3(c)).

Let M be a smooth mnfd, let $S \subseteq M$ be an immersed submnfd and let $\iota : S \hookrightarrow M$ be the inclusion map. If $\omega \in \mathfrak{X}^*(M)$, then $\iota^*\omega \in \mathfrak{X}^*(S)$. More precisely, given $p \in S$ and $v \in T_pS$, we have

$$(\iota^*\omega)_p v = \omega_p(d\iota_p(v)) = \omega_p(v),$$

since $d\iota_p : T_pS \hookrightarrow T_pM$ is just the inclusion map under our usual identification of T_pS with the subspace $d\iota_p(T_pS)$ of T_pM. Thus, $\iota^*\omega$ is just the restriction of ω to vectors tangent to S. For this reason, $\iota^*\omega$ is often called <u>the restriction of ω to S</u>. Note, however, that $\iota^*\omega$ might equal zero at a given pt of S, even though considered as a covector field on M, ω might not vanish there. For example:

<u>EXAMPLE 8.14</u>: Consider $\omega = dy \in \mathfrak{X}^*(\mathbb{R}^2)$ and let $S : (y=0)$ be the x-axis, considered as an embedded submnfd of $\mathbb{R}^2$. As a covector field on $\mathbb{R}^2$, ω is clearly nonzero everywhere, because one of its components is always equal to 1. However, the restriction $\iota^*\omega$ of ω to S is identically zero, because y vanishes identically on S:

$$\iota^*\omega = \iota^*dy = d(y \circ \iota) = 0.$$

To distinguish the two ways in which we might interpret the statement "ω vanishes on S", one usually says that

ω vanishes along S (or vanishes at pts of S) if $\omega_p = 0$ for every $p \in S$. The weaker condition that $\iota^* \omega = 0$ is expressed by saying that the restriction of ω to S vanishes (or the pullback of ω to S vanishes).