

# Differential Geometry II - Smooth Manifolds Winter Term 2023/2024 Lecturer: Dr. N. Tsakanikas Assistant: L. E. Rösler

# Exercise Sheet 12

## Exercise 1:

- (a) Restricting smooth vector fields to submanifolds: Let M be a smooth manifold, let S be an immersed submanifold of M, and let  $\iota: S \hookrightarrow M$  be the inclusion map. Prove the following assertions:
  - (i) If  $Y \in \mathfrak{X}(M)$  and if there is  $X \in \mathfrak{X}(S)$  that is *i*-related to Y, then  $Y \in \mathfrak{X}(M)$  is tangent to S.
  - (ii) If Y ∈ 𝔅(M) is tangent to S, then there is a unique smooth vector field on S, denote by Y|<sub>S</sub>, which is *ι*-related to Y.
    [Hint: Determine first the candidate vector field on S and then use *Theorem 5.6* and *Proposition 5.16* to show that it is smooth.]
- (b) Lie brackets of smooth vector fields tangent to submanifolds: Let M be a smooth manifold and let S be an immersed submanifold of M. If  $Y_1$  and  $Y_2$  are smooth vector fields on M that are tangent to S, then show that their Lie bracket  $[Y_1, Y_2]$  is also tangent to S.

### Exercise 2:

Let V be a smooth vector field on a smooth manifold M, let  $J \subseteq \mathbb{R}$  be an interval, and let  $\gamma: J \to M$  be an integral curve of V. Prove the following assertions:

(a) Rescaling lemma: For any  $a \in \mathbb{R}$ , the curve

$$\widetilde{\gamma} \colon \widetilde{J} \to M, \ t \mapsto \gamma(at)$$

is an integral curve of the vector field  $\widetilde{V} := aV$  on M, where  $\widetilde{J} := \{t \in \mathbb{R} \mid at \in J\}$ .

(b) Translation lemma: For any  $b \in \mathbb{R}$ , the curve

$$\widehat{\gamma} \colon \widehat{J} \to M, \ t \mapsto \gamma(t+b)$$

is also an integral curve of V on M, where  $\widehat{J} := \{t \in \mathbb{R} \mid t+b \in J\}.$ 

## Exercise 3 (to be submitted by Friday, 15.12.2023, 20:00):

(a) Compute the Lie bracket [X, Y] of each of the following pairs of smooth vector fields X and Y on  $\mathbb{R}^3$ :

(i) 
$$X = y \frac{\partial}{\partial z} - 2xy^2 \frac{\partial}{\partial y}$$
 and  $Y = \frac{\partial}{\partial y}$ .  
(ii)  $X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$  and  $Y = -z \frac{\partial}{\partial y} + y \frac{\partial}{\partial z}$ 

(b) Compute the flow of each of the following smooth vector fields on  $\mathbb{R}^2$ :

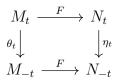
(i) 
$$V = x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}$$
.  
(ii)  $W = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$ .

#### Exercise 4:

Let  $\theta \colon \mathbb{R} \times M \to M$  be a smooth global flow on a smooth manifold M. Show that the infinitesimal generator V of  $\theta$  is a smooth vector field on M, and that each curve  $\theta^{(p)} \colon \mathbb{R} \to M$  is an integral curve of V.

#### Exercise 5:

(a) Naturality of flows: Let  $F: M \to N$  be a smooth map. Let  $X \in \mathfrak{X}(M)$  and  $Y \in \mathfrak{X}(N)$ . Let  $\theta$  be the flow of X and  $\eta$  be the flow of Y. Show that if X and Y are F-related, then for each  $t \in \mathbb{R}$  it holds that  $F(M_t) \subseteq N_t$  and  $\eta_t \circ F = F \circ \theta_t$  on  $M_t$ :



(b) Diffeomorphism invariance of flows: Let  $F: M \to N$  be a diffeomorphism. If  $X \in \mathfrak{X}(M)$  and  $\theta$  is the flow of X, then show that the flow of  $F_*X$  is  $\eta_t = F \circ \theta_t \circ F^{-1}$ , with domain  $N_t = F(M_t)$  for each  $t \in \mathbb{R}$ .

**Definition.** Let V be a (rough) vector field on a smooth manifold M. A point  $p \in M$  is called a *singular point* of V if  $V_p = 0 \in T_pM$ ; otherwise, it is called a *regular point* of V.

### Exercise 6:

Let V be a smooth vector field on a smooth manifold M and let  $\theta \colon \mathfrak{D} \to M$  be the flow generated by V. Prove the following assertions:

- (a) If  $p \in M$  is a singular point of V, then  $\mathfrak{D}^{(p)} = \mathbb{R}$  and  $\theta^{(p)}$  is the constant curve  $\theta^{(p)}(t) \equiv p$ .
- (b) If  $p \in M$  is a regular point of V, then  $\theta^{(p)} \colon \mathfrak{D}^{(p)} \to M$  is a smooth immersion.

[Hint: Argue by contraposition and use the fundamental theorem on flows.]