Differential Geometry II - Smooth Manifolds
Winter Term 2023/2024
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Exercise Sheet 11 – Solutions

Exercise 1 (*Uniqueness of the smooth structure on TM*)**:** Let M be a smooth n-manifold. Show that the topology and smooth structure on the tangent bundle TM constructed in *Proposition 3.12* are the unique ones with respect to which $\pi\colon TM \to M$ is a smooth vector bundle with the given vector space structure on the fibers, and such that all coordinate vector fields are smooth local sections.

[Hint: Use *Exercise* 5*, Sheet* 10.]

Solution: Denote by $(TM)'$ a smooth manifold with underlying set $TM = \bigsqcup_{p\in M} T_pM$, but with possibly different topology and smooth structure than the usual tangent bundle TM, such that the map $\pi'\colon (TM)' \to M$, sending $v \in T_pM$ to $p \in M$, gives $(TM)'$ the structure of a smooth vector bundle over M such that all the coordinate vector fields are smooth local sections (note that, as set-theoretic maps, π and π' are the same, but we denote them differently to emphasize that their source may have different topology and smooth structure). By assumption, if $\big(U, (x^1, \dots, x^n)\big)$ is a smooth chart for M, then the maps

$$\frac{\partial}{\partial x^i}\colon U \to (TM)'$$
$$p \mapsto \left.\frac{\partial}{\partial x^i}\right|_p$$

are smooth local sections of π'. Since for every $p \in U$ the vectors $\left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p$ form a basis of T_pM, it follows that $\left(\left.\frac{\partial}{\partial x^i}\right|_p\right)_{1\le i\le n}$ is a smooth local frame for $(TM)'$, and according to part (d) of *Exercise* 5*, Sheet* 10, the map

$$\widetilde{\varphi}\colon (\pi')^{-1}(U) \to \varphi(U) \times \mathbb{R}^n$$
$$\left(p, v^i \left.\frac{\partial}{\partial x^i}\right|_p\right) \mapsto \big(\varphi(p), v^1, \dots, v^n\big)$$

is a smooth chart for $(TM)'$. But the same holds for TM (as we saw in the proof of *Proposition 3.12*). It follows that the identity map $TM \to (TM)'$ is a diffeomorphism. In particular, it is a homeomorphism, and thus also the topology agrees.

Remark. We somewhat used in *Exercise* 1 that the smooth structure actually determines the topology. That is, we have the following:

Let M be a set and let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on M, both endowing it with the structure of a topological manifold. Supposed that $\mathcal{A}$ is an atlas for both topologies, such that both $(M, \mathcal{T}, \mathcal{A})$ and $(M, \mathcal{T}', \mathcal{A})$ are smooth manifolds. Then $\mathcal{T} = \mathcal{T}'$. Indeed, the identity $\mathrm{Id}_M \colon (M, \mathcal{T}, \mathcal{A}) \to (M, \mathcal{T}', \mathcal{A})$ is smooth, as we have the same atlas on both sides; in particular, it is continuous, so $\mathcal{T}' \subseteq \mathcal{T}$. A symmetric argument also shows that the reverse inclusion holds. Therefore, $\mathcal{T} = \mathcal{T}'$, as claimed.

Exercise 2:

(a) Consider the tangent bundle $\pi \colon T\mathbb{S}^2 \to \mathbb{S}^2$ of the unit sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$. Compute the transition function associated with the two local trivializations determined by stereographic coordinates.

(b) Show that there is a smooth vector field on $\mathbb{S}^2$ which vanishes at exactly one point.

[Hint: Use the stereographic projection and consider one of the coordinate vector fields.]

Solution:

(a) We use the same notation as the one used in *Exercise* 6*, Sheet* 2. According to (the solution of) part (c) of *Exercise* 1*, Sheet* 10, the transition function

$$\tau \colon \mathbb{S}^2 \setminus \{N, S\} \to \mathrm{GL}(\mathbb{R}, 2)$$

between the two smooth local trivializations of $T\mathbb{S}^2$ determined by the stereographic coordinates $\big(\mathbb{S}^2 \setminus \{N\}, \sigma\big)$ and $\big(\mathbb{S}^2 \setminus \{S\}, \widetilde{\sigma}\big)$ is given at every point $p \in \mathbb{S}^2 \setminus \{N, S\}$ by the Jacobian matrix at $\widehat{p} = \sigma(p) = \widetilde{\sigma}(p)$ of the transition map $\sigma \circ \widetilde{\sigma}^{-1}$. We have seen in *Exercise* 6*, Sheet* 2 that $\sigma \circ \widetilde{\sigma}^{-1}$ is given by the formula

$$\big(\sigma \circ \widetilde{\sigma}^{-1}\big)(\widetilde{u}, \widetilde{v}) = \left(\frac{\widetilde{u}}{\widetilde{u}^2 + \widetilde{v}^2}, \frac{\widetilde{v}}{\widetilde{u}^2 + \widetilde{v}^2}\right) = (u, v), \ \ (\widetilde{u}, \widetilde{v}) \in \mathbb{R}^2 \setminus \{(0,0)\},$$

and thus its Jacobian at an arbitrary point $(\widetilde{u}, \widetilde{v}) \in \mathbb{R}^2 \setminus \{(0,0)$ is the matrix

$$J\big(\sigma \circ \widetilde{\sigma}^{-1}\big)(\widetilde{u}, \widetilde{v}) = \begin{pmatrix} \frac{\widetilde{v}^2 - \widetilde{u}^2}{(\widetilde{u}^2 + \widetilde{v}^2)^2} & \frac{-2\widetilde{u}\widetilde{v}}{(\widetilde{u}^2 + \widetilde{v}^2)^2} \\ \frac{-2\widetilde{u}\widetilde{v}}{(\widetilde{u}^2 + \widetilde{v}^2)^2} & \frac{\widetilde{u}^2 - \widetilde{v}^2}{(\widetilde{u}^2 + \widetilde{v}^2)^2} \end{pmatrix}$$

(whose determinant equal to $-\dfrac{1}{(\widetilde{u}^2 + \widetilde{v}^2)^2}$, which is clearly non-zero).

(b) We view (u, v), resp. $(\widetilde{u}, \widetilde{v})$, as the component functions of σ, resp. $\widetilde{\sigma}$, where

$$\sigma(x, y, z) = \left(\frac{x}{1 - z}, \frac{y}{1 - z}\right) = (u, v) \quad \text{and} \quad \widetilde{\sigma}(x, y, z) = \left(\frac{x}{1 + z}, \frac{y}{1 + z}\right) = (\widetilde{u}, \widetilde{v}),$$

so that

$$u = u(\widetilde{u}, \widetilde{v}) = \frac{\widetilde{u}}{\widetilde{u}^2 + \widetilde{v}^2} \quad \text{and} \quad v = v(\widetilde{u}, \widetilde{v}) = \frac{\widetilde{v}}{\widetilde{u}^2 + \widetilde{v}^2},$$

resp.

$$\widetilde{u} = \widetilde{u}(u,v) = \frac{u}{u^2+v^2} \qquad \text{and} \qquad \widetilde{v} = \widetilde{v}(u,v) = \frac{v}{u^2+v^2}.$$

Note that $\widetilde{\sigma} \circ \sigma^{-1}$ is given essentially by the same formula as $\sigma \circ \widetilde{\sigma}^{-1}$ (with the roles of (u,v) and $(\widetilde{u},\widetilde{v})$ reversed), and thus its Jacobian is essentially the same matrix as the one in part (a) (where everything is now expressed in terms of (u,v) instead of $(\widetilde{u},\widetilde{v})$); see *Exercise* 6, *Sheet* 2.

We now consider the first coordinate vector field $X := \frac{\partial}{\partial u}$ associated with the chart $\big(\mathbb{S}^2 \setminus \{N\}, \sigma\big)$ for $\mathbb{S}^2$. It follows from *Proposition 7.3* that $X = 1\frac{\partial}{\partial u} + 0\frac{\partial}{\partial v}$ is a smooth vector field on $\mathbb{S}^2 \setminus \{N\}$, since its component functions with respect to the smooth coordinate frame $\big\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\big\}$ are constant, and it is obvious that X does not vanish on $\mathbb{S}^2 \setminus \{N\}$. We claim that X extends to a smooth vector field on the whole $\mathbb{S}^2$ and that it vanishes precisely at the north pole $N \in \mathbb{S}^2$. Indeed, on $\mathbb{S}^2 \setminus \{N, S\}$ we have

$$\begin{aligned} X &= \frac{\partial \widetilde{u}}{\partial u}\frac{\partial}{\partial \widetilde{u}} + \frac{\partial \widetilde{v}}{\partial u}\frac{\partial}{\partial \widetilde{v}} = \frac{v^2-u^2}{(u^2+v^2)^2}\frac{\partial}{\partial \widetilde{u}} + \frac{-2uv}{(u^2+v^2)^2}\frac{\partial}{\partial \widetilde{v}} \\ &= \big(\widetilde{v}^2 - \widetilde{u}^2\big)\frac{\partial}{\partial \widetilde{u}} + \big(-2\widetilde{u}\widetilde{v}\big)\frac{\partial}{\partial \widetilde{v}}. \end{aligned}$$

Since $N = (0,0,1) \in \mathbb{S}^2$ corresponds under $\widetilde{\sigma}$ to $(\widetilde{u},\widetilde{v}) = (0,0) \in \mathbb{R}^2$, we infer that X can be extended to a vector field on $\mathbb{S}^2$ by defining its value at N to be zero; namely,

$$X\colon \mathbb{S}^2 \to T\mathbb{S}^2,\ p \mapsto \begin{cases} \frac{\partial}{\partial u}\big|_p, & \text{if } p \neq N, \\ 0, & \text{if } p = N. \end{cases}$$

The above expression for X also shows that its component functions with respect to the smooth coordinate frame $\big\{\frac{\partial}{\partial \widetilde{u}}, \frac{\partial}{\partial \widetilde{v}}\big\}$ associated with the chart $\big(\mathbb{S}^2 \setminus \{S\}, \widetilde{\sigma}\big)$ are smooth, and hence X is smooth (also) on $\mathbb{S}^2 \setminus \{S\}$ by *Proposition 7.3*. Therefore, X is a smooth vector field on $\mathbb{S}^2$ which vanishes only at the north pole N of $\mathbb{S}^2$, as claimed.

Exercise 3: Consider the *Euler vector field* on $\mathbb{R}^n$, i.e., the vector field V on $\mathbb{R}^n$ whose value at a point $x = (x^1, \dots, x^n) \in \mathbb{R}^n$ is

$$V_x = x^1 \frac{\partial}{\partial x_1}\bigg|_x + \ldots + x^n \frac{\partial}{\partial x_n}\bigg|_x.$$

(a) Check that V is a smooth vector field on $\mathbb{R}^n$.

(b) Let $c \in \mathbb{R}$ and let $f\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be a smooth function which is *positively homogeneous of degree* c, i.e., $f(\lambda x) = \lambda^c f(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$. Prove that $Vf = cf$.

[Hint: Differentiate the above relation with respect to both x^i and λ.]

(c) Compute the integral curves of V.

Solution:

(a) Note that the component functions of V with respect to the standard coordinate frame for $\mathbb{R}^n$ are linear, hence smooth. Therefore, V is a smooth vector field on $\mathbb{R}^n$ by *Proposition 7.3*.

(b) Using the chain rule, we obtain

$$\lambda^c \frac{\partial f}{\partial x^i}(x) = \frac{\partial}{\partial x^i}\big(\lambda^c f(x)\big) = \frac{\partial}{\partial x^i}\big(f(\lambda x)\big) = \lambda \frac{\partial f}{\partial x^i}(\lambda x) \tag{1}$$

and

$$c\lambda^{c-1} f(x) = \frac{d}{d\lambda}\big(\lambda^c f(x)\big) = \frac{d}{d\lambda}\big(f(\lambda x)\big) = \sum_{i=1}^n x^i \frac{\partial f}{\partial x^i}(\lambda x). \tag{2}$$

Since

$$(Vf)(x) = V_x f = \sum_{i=1}^n x^i \frac{\partial f}{\partial x^i}(x),$$

we have

$$(Vf)(\lambda x) = V_{\lambda x} f = \sum_{i=1}^n (\lambda x^i) \frac{\partial f}{\partial x^i}(\lambda x) \overset{(1)}{=} \lambda^c \sum_{i=1}^n x^i \frac{\partial f}{\partial x^i}(x) = \lambda^c (Vf)(x) \tag{3}$$

but also

$$(Vf)(\lambda x) = V_{\lambda x} f = \sum_{i=1}^n (\lambda x^i) \frac{\partial f}{\partial x^i}(\lambda x) \overset{(2)}{=} c\lambda^c f(x). \tag{4}$$

It follows now from (3) and (4) that

$$(Vf)(x) = cf(x) \text{ for every } x \in \mathbb{R}^n \setminus \{(0,0)\},$$

as desired.

(c) Since at $p = (0, \ldots, 0) \in \mathbb{R}^n$ we have $V_p = (0, \ldots, 0)$, the unique maximal integral curve of V starting at p is the constant curve

$$\gamma_0 \colon \mathbb{R} \to \mathbb{R}^n,\ t \mapsto (0, \ldots, 0).$$

Now, if $\gamma \colon J \to \mathbb{R}^n$ is a smooth curve, written in standard coordinates as $\gamma(t) = \big(\gamma^1(t), \ldots, \gamma^n(t)\big)$, then the condition $\gamma'(t) = V_{\gamma(t)}$ for γ to be an integral curve of V translates to

$$\dot{\gamma}^j(t) = \gamma^j(t) \ \text{ for every } 1 \le j \le n,$$

which yields

$$\gamma^j \colon J = \mathbb{R} \to \mathbb{R},\ \gamma^j(t) = c_j e^t, \quad 1 \le j \le n$$

for some constants $c_j \in \mathbb{R}$. Therefore, the unique maximal integral curve of V starting at $p = (p^1, \ldots, p^n) \in \mathbb{R}^n$ is the smooth curve

$$\gamma \colon \mathbb{R} \to \mathbb{R}^n,\ t \mapsto \big(p^1 e^t, \ldots, p^n e^t\big).$$

Finally, observe that the Euler vector field V is a *complete* vector field on $\mathbb{R}^n$.

Remark. The statement from *Exercise* 3(b) is referred to as *the Euler's homogeneous function theorem* in the literature. In fact, it can also be shown that the converse to Euler's homogeneous function theorem holds: *if* $f \in C^\infty\big(\mathbb{R}^n \setminus \{(0,0)\}\big)$ *satisfies* $Vf = cf$, *where* V *is the Euler vector field on* $\mathbb{R}^n$ *and* $c \in \mathbb{R}$, *then* f *is positively homogeneous of degree* c.

Exercise 4:

(a) Let $F\colon M \to N$ be a smooth map. Let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. Show that X and Y are F-related if and only if for every smooth real-valued function f defined on an open subset of N, we have

$$X(f \circ F) = (Yf) \circ F.$$

(b) Consider the smooth map

$$F\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (\cos t, \sin t)$$

and the smooth vector fields

$$X = \frac{d}{dt} \in \mathfrak{X}(\mathbb{R}) \quad \text{and} \quad Y = x\,\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^2).$$

Show that X and Y are F-related.

(c) Let $F\colon M \to N$ be a diffeomorphism and let $X \in \mathfrak{X}(M)$. Prove that there exists a unique smooth vector field Y on N that is F-related to X. The vector field Y is denoted by F_*X and is called the *pushforward of* X *by* F.

(d) Consider the open submanifolds

$$M := \big\{(x,y) \in \mathbb{R}^2 \mid y > 0 \ \text{ and } \ x + y > 0\big\}$$

and

$$N := \big\{(u,v) \in \mathbb{R}^2 \mid u > 0 \ \text{ and } \ v > 0\big\}$$

of $\mathbb{R}^2$ and the map

$$F\colon M \to N,\ (x,y) \mapsto \big(x + y, \frac{x}{y} + 1\big).$$

(i) Show that F is a diffeomorpism and compute its inverse F^{-1}.

(ii) Compute the pushforward F_*X of the following smooth vector field X on M:

$$X_{(x,y)} = y^2 \frac{\partial}{\partial x}\bigg|_{(x,y)}.$$

(e) *Naturality of integral curves*: Let $F\colon M \to N$ be a smooth map. Show that $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are F-related if and only if F takes integral curves of X to integral curves of Y.

Solution:

(a) For any point $p \in M$ and any smooth real-valued function f defined on an open neighborhood of $F(p)$ we have

$$X(f \circ F)(p) = X_p(f \circ F) = dF_p(X_p)(f)$$

and

$$\big((Yf) \circ F\big)(p) = (Yf)\big(F(p)\big) = Y_{F(p)}f.$$

Therefore, X and Y are F-related, i.e., $dF_p(X_p) = Y_{F(p)}$ for every $p \in M$, if and only if for every smooth real-valued function f defined on an open subset of N it holds that $X(f \circ F) = (Yf) \circ F$.

(b) *1st way*: We first prove the claim using the definition of F-related vector fields. To this end, recall that the differential of F at an arbitrary point $t \in \mathbb{R}$ is represented (with respect to the bases $\{d/dt\big|_t\}$ for $T_t\mathbb{R} \cong \mathbb{R}$ and $\{\partial/\partial x\big|_{F(t)}, \partial/\partial y\big|_{F(t)}\}$ for $T_{F(t)}\mathbb{R}^2 \cong \mathbb{R}^2$) by the Jacobian of F at t, which is the 2×1-matrix

$$\begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}.$$

Hence,

$$dF_t(X_t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} \cdot (1) = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} = -\sin(t)\frac{\partial}{\partial x}\bigg|_{F(t)} + \cos(t)\frac{\partial}{\partial y}\bigg|_{F(t)} = Y_{F(t)}$$

for any $t \in \mathbb{R}$, which shows that X and Y are F-related.

2nd way: Alternatively, we may also prove the assertion using (a) as follows: For every smooth real-valued function $f = f(x, y)$ defined on an open subset of $\mathbb{R}^2$ and for any $t \in \mathbb{R}$ we have

$$\begin{aligned} X(f \circ F)(t) = X_t(f \circ F) &= \frac{d}{dt}\bigg|_t (f \circ F) \\ &= \left(\frac{\partial f}{\partial x}\big(F(t)\big),\ \frac{\partial f}{\partial y}\big(F(t)\big)\right) \cdot \big(F_1'(t),\ F_2'(t)\big)^T \\ &= -\sin(t)\,\frac{\partial f}{\partial x}\big(F(t)\big) + \cos(t)\,\frac{\partial f}{\partial y}\big(F(t)\big) \end{aligned}$$

and

$$\begin{aligned} \big((Yf) \circ F\big)(t) = (Yf)\big(F(t)\big) &= Y_{F(t)}f \\ &= \left(\cos(t)\,\frac{\partial}{\partial y}\bigg|_{F(t)} - \sin(t)\,\frac{\partial}{\partial x}\bigg|_{F(t)}\right) f \\ &= \cos(t)\,\frac{\partial f}{\partial y}\big(F(t)\big) - \sin(t)\,\frac{\partial f}{\partial x}\big(F(t)\big), \end{aligned}$$

so part (a) implies that X and Y are F-related.

(c) Since F is a diffeomorphism, for every $p \in M$ its differential $dF_p \colon T_pM \to T_{F(p)}N$ is an $\mathbb{R}$-linear isomorphism by part (d) of *Exercise* 1, *Sheet* 4. We may thus define the following rough vector field on N:

$$Y \colon N \to TN,\ q \mapsto dF_{F^{-1}(q)}\big(X_{F^{-1}(q)}\big),$$

and it is clear that this is the unique (rough) vector field on N that is F-related to X. We now observe that Y is the composition of the following smooth maps:

$$N \xrightarrow{F^{-1}} M \xrightarrow{X} TM \xrightarrow{dF} TN,$$

and hence it is smooth by part (e) of *Exercise* 3, *Sheet* 3 (see also part (a) of *Exercise* 4, *Sheet* 5).

Remark. Given a diffeomorphism $F \colon M \to N$, the pushforward of any $X \in \mathfrak{X}(M)$ by F is defined explicitly by the formula

$$(F_*X)_q = dF_{F^{-1}(q)}\big(X_{F^{-1}(q)}\big),$$

as already demonstrated in the proof of (c) above. As long as the inverse map F^{-1} of F can be computed explicitly, the pushforward of a smooth vector field can be computed directly from this formula. This observation will be applied in (d) below.

(d) It is straightforward to check that the inverse of F is given by the formula

$$F^{-1}(u,v) = \left(u - \frac{u}{v}, \frac{u}{v}\right).$$

The differential of F at an arbitrary point $(x,y) \in M$ is represented by the Jacobian of F at (x,y), given by

$$DF(x,y) = \begin{pmatrix} 1 & 1 \\ \frac{1}{y} & -\frac{x}{y^2} \end{pmatrix},$$

and thus $dF_{F^{-1}(u,v)}$ is represented by the matrix

$$DF\left(u - \frac{u}{v}, \frac{u}{v}\right) = \begin{pmatrix} 1 & 1 \\ \frac{v}{u} & \frac{v-v^2}{u} \end{pmatrix}.$$

For any $(u,v) \in N$ we have

$$X_{F^{-1}(u,v)} = X_{\left(u-\frac{u}{v},\frac{u}{v}\right)} = \frac{u^2}{v^2}\frac{\partial}{\partial x}\bigg|_{\left(u-\frac{u}{v},\frac{u}{v}\right)}.$$

Therefore, we obtain

$$(F_*X)_{(u,v)} = \frac{u^2}{v^2}\frac{\partial}{\partial u}\bigg|_{(u,v)} + \frac{u}{v}\frac{\partial}{\partial v}\bigg|_{(u,v)}.$$

(e) Assume first that X and Y are F-related. Let γ be an integral curve of X. By definition and by part (b) of *Exercise* 5, *Sheet* 4 we obtain

$$(F \circ \gamma)'(t) = dF_{\gamma(t)}\big(\gamma'(t)\big) = dF_{\gamma(t)}\big(X_{\gamma(t)}\big) = Y_{F(\gamma(t))} = Y_{(F\circ\gamma)(t)},$$

which shows that $F \circ \gamma$ is an integral curve of Y.

Assume now that F takes integral curves of X to integral curves of Y. Let $p \in M$ and let $\gamma\colon (-\varepsilon, \varepsilon) \to M$ be an integral curve of X starting at p. Then $\gamma(0) = p$ and $\gamma'(0) = X_p$. Moreover, by assumption, $F \circ \gamma\colon (-\varepsilon, \varepsilon) \to N$ is an integral curve of Y starting at $F(p)$, so $Y_{(F\circ\gamma)(0)} = (F \circ \gamma)'(0)$. Therefore, by part (b) of *Exercise* 5, *Sheet* 4 we obtain

$$Y_{F(p)} = (F \circ \gamma)'(0) = dF_p\big(\gamma'(0)\big) = dF_p\big(X_p\big).$$

Since $p \in M$ was arbitrary, we conclude that X and Y are F-related.

Exercise 5:

Let M be a smooth n-manifold and let X and Y be smooth vector fields on M.

(a) *Coordinate formula for the Lie bracket*: Let

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i} \quad \text{and} \quad Y = \sum_{j=1}^{n} Y^j \frac{\partial}{\partial x^j}$$

be the coordinate expressions for X and Y, respectively, in terms of some smooth local coordinates (x^i) for M. Show that the Lie bracket $[X, Y]$ has the following coordinate expression:

$$[X, Y] = \sum_{j=1}^{n} \sum_{i=1}^{n} \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

(b) Compute the Lie brackets $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]$ of the coordinate vector fields $\partial/\partial x^i$ in any smooth chart $\big(U, (x^i)\big)$ for M.

(c) Assume now that $M = \mathbb{R}^3$,

$$X = x\,\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x(y+1)\,\frac{\partial}{\partial z} \quad \text{and} \quad Y = \frac{\partial}{\partial x} + y\,\frac{\partial}{\partial z},$$

and compute the Lie bracket $[X, Y]$.

Solution:

(a) For any $f \in C^\infty(M)$ we have

$$\begin{aligned}
[X,Y](f) = XY(f) - YX(f) &= X\left(\sum_j Y^j \frac{\partial f}{\partial x^j}\right) - Y\left(\sum_i X^i \frac{\partial f}{\partial x^i}\right) \\
&= \sum_j \left[X(Y^j)\frac{\partial f}{\partial x^j} + Y^j X\left(\frac{\partial f}{\partial x^j}\right)\right] - \sum_i \left[Y(X^i)\frac{\partial f}{\partial x^i} - X^i Y\left(\frac{\partial f}{\partial x^i}\right)\right] \\
&= \sum_{j,i} \left[X^i \frac{\partial Y^j}{\partial x^i}\frac{\partial f}{\partial x^j} + Y^j X^i \frac{\partial}{\partial x^i}\left(\frac{\partial f}{\partial x^j}\right)\right] - \sum_{i,j} \left[Y^j \frac{\partial X^i}{\partial x^j}\frac{\partial f}{\partial x^i} - X^i Y^j \frac{\partial}{\partial x^j}\left(\frac{\partial f}{\partial x^i}\right)\right] \\
&= \sum_{i,j} \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}\right)\frac{\partial f}{\partial x^j} - \sum_{i,j} (X^i Y^j) \underbrace{\left[\frac{\partial}{\partial x^i}\left(\frac{\partial f}{\partial x^j}\right) - \frac{\partial}{\partial x^j}\left(\frac{\partial f}{\partial x^i}\right)\right]}_{=0} \\
&= \sum_{i,j} \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}\right)\frac{\partial f}{\partial x^j}
\end{aligned}$$

(b) Recall that the component functions of each coordinate vector field $\partial/\partial x^j$ in the coordinate frame $(\partial/\partial x^i)$ associated with the smooth chart $\big(U,(x^i)\big)$ are constant, so it follows immediately from (a) that

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0.$$

(c) By part (a) we obtain

$$\begin{aligned}[X,Y] = &\big((x\cdot 0 - 1\cdot 1) + (1\cdot 0 - 0\cdot 0) + (x(y+1)\cdot 0 - 1\cdot 0)\big)\frac{\partial}{\partial x}\\ &+ \big((x\cdot 0 - 1\cdot 0) + (x\cdot 0 - 0\cdot 0) + (x(y+1)\cdot 0 - y\cdot 0)\big)\frac{\partial}{\partial y}\\ &+ \big((x\cdot 0 - 1\cdot (y+1)) + (1\cdot 1 - 0\cdot x) + (x(y+1)\cdot 0 - y\cdot 0)\big)\frac{\partial}{\partial z}\\ = &-\frac{\partial}{\partial x} - y\frac{\partial}{\partial z}.\end{aligned}$$

Exercise 6 (*Properties of the Lie bracket*)**:** Let M be a smooth manifold. Show that the Lie bracket satisfies the following identities for all $X,Y,Z\in\mathfrak{X}(M)$:

(a) *Bilinearity*: For all $a,b\in\mathbb{R}$ we have

$$\begin{aligned}[aX+bY,Z] &= a[X,Z]+b[Y,Z],\\ [Z,aX+bY] &= a[Z,X]+b[Z,Y].\end{aligned}$$

(b) *Antisymmetry*:

$$[X,Y] = -[Y,X].$$

(c) *Jacobi identity*:

$$\big[X,[Y,Z]\big] + \big[Y,[Z,X]\big] + \big[Z,[X,Y]\big] = 0.$$

(d) For all $f,g\in C^\infty(M)$ we have

$$[fX,gY] = fg[X,Y] + (fXg)Y - (gYf)X.$$

Solution:

(a) We first make the following observation: given $\lambda,\mu\in\mathbb{R}$ and $U,V,W\in\mathfrak{X}(M)$, for any $f\in C^\infty(M)$ it holds that

$$(\lambda V+\mu W)\,Uf = \lambda VUf + \mu WUf \quad\text{and}\quad U(\lambda V+\mu W)f = \lambda UVf + \mu UWf.$$

Indeed, for any $p\in M$ we have

$$\begin{aligned}\big((\lambda V+\mu W)\,Uf\big)(p) &= (\lambda V+\mu W)_p(Uf) = (\lambda V_p + \mu W_p)(Uf)\\ &= \lambda V_p(Uf) + \mu W_p(Uf) = \lambda V(Uf)(p) + \mu W(Uf)(p)\\ &= \big(\lambda VUf + \mu WUf\big)(p),\end{aligned}$$

which yields the first equality above, while the second one is obtained analogously.

Now, given $a, b \in \mathbb{R}$, using the previous observation, for any $f \in C^\infty(M)$ we have

$$\begin{aligned}[aX + bY, Z](f) &= \big(aX + bY\big)Zf - Z\big(aX + bY\big)f \\ &= aXZf + bYZf - aZXf - bZYf \\ &= a\big(XZf - ZXf\big) + b\big(YZf - ZYf\big) \\ &= a[X, Z](f) + b[Y, Z](f) \\ &= \big(a[X, Z] + b[Y, Z]\big)(f),\end{aligned}$$

which yields the first part of the statement, while the second one is obtained similarly.

(b) For any $f \in C^\infty(M)$ we have

$$[X, Y](f) = XYf - YXf = -\big(YXf - XYf\big) = -[Y, X](f),$$

which yields the statement.

(c) By expanding all the brackets and using linearity we obtain

$$\begin{aligned}&\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = \\ &\quad = X[Y, Z] - [Y, Z]X + Y[Z, X] - [Z, X]Y + Z[X, Y] - [X, Y]Z \\ &\quad = XYZ - XZY - YZX + ZYX + YZX - YXZ - ZXY + XZY + \\ &\qquad + ZXY - ZYX - XYZ + YXZ \\ &\quad = 0.\end{aligned}$$

(d) We first make the following observation: if $V \in \mathfrak{X}(M)$ and $s, t \in C^\infty(M)$, then

$$(sV)\,h = s\,(Vh) \quad \text{(as smooth functions on } M\text{)},$$

since for any $p \in M$ we have

$$\big((sV)\,h\big)(p) = (sV)_p\,h = \big(s(p)V_p\big)h = s(p)V_ph = s(p)(Vh)(p) = \big(s\,(Vh)\big)(p).$$

Now, fix $f, g \in C^\infty(M)$. Using the previous observation and the fact that smooth vector fields are derivations of $C^\infty(M)$ by *Proposition 7.6*, for any $h \in C^\infty(M)$ we have

$$\begin{aligned}[fX, gY](h) &= (fX)(gY)(h) - (gY)(fX)(h) \\ &= (fX)\big(g(Yh)\big) - (gY)\big(f(Xh)\big) \\ &= g(fX)(Yh) + (Yh)(fX)(g) - f(gY)(Xh) - (Xh)(gY)(f) \\ &= gf\big(X(Yh)\big) + f(Xg)(Yh) - fg\big(Y(Xh)\big) - g(Yf)(Xh) \\ &= fg\big((XY - YX)(h)\big) + (fXg)Y(h) - (gYf)X(h) \\ &= \big(fg[X, Y] + (fXg)Y - (gYf)X\big)(h),\end{aligned}$$

whence the desired relation.

Remark. A *Lie algebra over* $\mathbb{R}$ is an $\mathbb{R}$-vector space $\mathfrak{g}$ endowed with a map $\mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$, called the *Lie bracket* and usually denoted by $(X, Y) \mapsto [X, Y]$, which satisfies the following properties for all $X, Y, Z \in \mathfrak{g}$:

(a) *Bilinearity*: For all $a, b \in \mathbb{R}$ we have

$$[aX + bY, Z] = a[X, Z] + b[Y, Z],$$
$$[Z, aX + bY] = a[Z, X] + b[Z, Y].$$

(b) *Antisymmetry*:

$$[X, Y] = -[Y, X].$$

(c) *Jacobi identity*:

$$\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = 0.$$

According to *Exercise* 6(a)(b)(c), the infinite-dimensional $\mathbb{R}$-vector space $\mathfrak{X}(M)$ of all smooth vector fields on a smooth manifold M is a Lie algebra under the Lie bracket. Another example of a Lie algebra is the $\mathbb{R}$-vector space $M_n(\mathbb{R})$ of real $n \times n$ matrices equipped with the *commutator bracket* $[A, B] \coloneqq AB - BA$.

Exercise 7: Let $F \colon M \to N$ be a smooth map.

(a) *Naturality of the Lie bracket*: Let $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(N)$ be vector fields such that X_i is F-related to Y_i for $i \in \{1, 2\}$. Show that $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$.

(b) *Pushforwards of Lie brackets*: Assume that F is a diffeomorphism and consider $X_1, X_2 \in \mathfrak{X}(M)$. Show that

$$F_*[X_1, X_2] = [F_*X_1, F_*X_2]\,.$$

Solution:

(a) Since X_i is F-related to Y_i for $i \in \{1, 2\}$, by *Exercise* 4(a) we infer that for every smooth real-valued function f defined on an open subset of N we have

$$X_1(f \circ F) = (Y_1 f) \circ F \quad \text{and} \quad X_2(f \circ F) = (Y_2 f) \circ F.$$

Therefore,

$$\begin{aligned}
[X_1, X_2](f \circ F) &= X_1 X_2(f \circ F) - X_2 X_1(f \circ F) \\
&= X_1\big((Y_2 f) \circ F\big) - X_2\big((Y_1 f) \circ F\big) \\
&= \big(Y_1(Y_2 f)\big) \circ F - \big(Y_2(Y_1 f)\big) \circ F \\
&= \big([Y_1, Y_2](f)\big) \circ F,
\end{aligned}$$

and thus *Exercise* 4(a) implies that $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$.

(b) Follows immediately from part (a) and from *Exercise* 4(c).