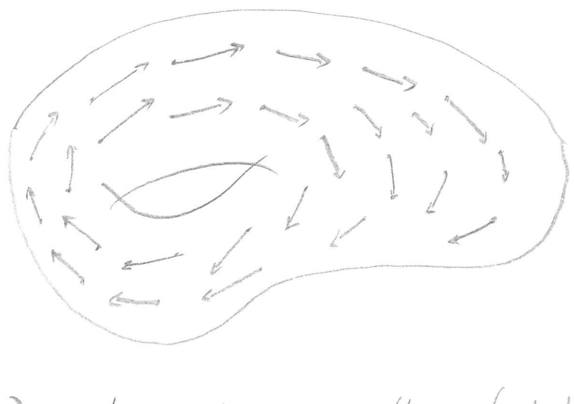
CH. 7: VECTOR FIELDS AND FLOWS

· DEF. 7.1: A rough/cont./smooth vector field on a smooth mnfd M is a rough/cont./smooth (global) section of the tangent bundle TM.

If $U \subseteq M$ is open, the fact that T_pU is naturally identified with T_pM for each $p \in U$ (PROP. 3.9) allows us to identify TU with the open subset $\pi^{-1}(U) \subseteq TM$. Therefore, a vector field on U can be thought of either as a map $U \to TU$ or as a map $U \to TM$.

A vector field on an open subset $U \subseteq \mathbb{R}^n$ is simply a cont. map $U \to \mathbb{R}^n$, which can be visualized as attaching an "arrow" to each pt of U. We visualize a vector field on an open subset U of a smooth mnfd M in a similar way: as an arrow attached to each pt of M, chosen to be tangent to M and to vary continuously from pt to pt.



The set $\mathfrak{X}(M)$ of all smooth (global) vector fields on a smooth mnfd M is an infinite-dimensional $\mathbb{R}$-v.s. and a module over the ring $C^\infty(M)$ (ES1OE3).

⇝ ∃ extension lemma for vector fields : special case of LEM. 6.10 ; see also ES10E3(d) for an application.

⇝ local / global frame for M = local / global frame for TM ; see DEF. 6.11.

⇝ completion of smooth local frames for M : special case of ES10E4

· DEF. 7.2 : Let M be a smooth mnfd and let $X : M \longrightarrow TM$ be a rough vector field on M. The support of M is defined as the closure of the set $\{ p \in M \mid X_p \neq 0 \}$. We say that X is compactly supported if its support is a compact set.

Let M and X be as above. If $(U, (x^i))$ is a smooth coordinate chart for M, then we can write the value of X at any pt $p \in U$ in terms of the coordinate basis vectors:

$$X_p = X^i(p) \left. \frac{\partial}{\partial x^i} \right|_p .$$

This defines n fncts $X^i : U \longrightarrow \mathbb{R}$, called the component fncts of X in the given chart.

· PROP. 7.3 (Smoothness criterion for vector fields) : Let M be a smooth mnfd and let $X : M \longrightarrow TM$ be a rough vector field. If $(U, (x^i))$ is any smooth coordinate chart on M, then the restriction of X to U is smooth iff its components fncts w.r.t. this chart are smooth.

PROOF: $(U,(x^i))$: smooth chart for M $\rightsquigarrow$
$\underset{=\varphi}{}$

$\rightsquigarrow$ $(\pi^{-1}(U), \underbrace{(x^i; v^i)}_{=\tilde\varphi})$: natural coordinates on TM

$\rightsquigarrow$ the coordinate repr. $\hat X$ of X w.r.t. these charts is

$$\hat X(x^1,\dots,x^n) = \tilde\varphi\left(X^i(\varphi^{-1}(x)) \frac{\partial}{\partial x^i}\Big|_{\varphi^{-1}(x)}\right)$$
$$= \left(x^1,\dots,x^n, X^1(\varphi^{-1}(x)),\dots,X^n(\varphi^{-1}(x))\right)$$

so X is smooth on U iff its component fncts X^i, $1 \le i \le n$ are smooth on U. ■

· EXAMPLE 7.4:

1) If $(U,(x^i))$ is any smooth chart on M, then the assignement $p \mapsto \frac{\partial}{\partial x^i}\Big|_p$ determines a vector field on U, called the i-th coordinate vector field and denoted by $\frac{\partial}{\partial x^i}$. It is smooth by Prop. 7.3, because its component fncts are constants.

In particular, the coordinate vector fields form a smooth local frame $\left(\frac{\partial}{\partial x^i}\right)$ for TM, called a coordinate frame. Note that every pt of M is in the domain of such a local frame.

2) The Euler vector field on $\mathbb{R}^n$; see ES11E3.

(More examples will appear later and in ES11, ES12, too.)

An essential property of vector fields is that they define operators on the space of smooth real-valued fncts. If $X \in \mathfrak{X}(M)$ and $f \in C^\infty(U)$, where $U \subseteq M$ is open, we obtain a new fnct $Xf : U \to \mathbb{R}$, defined by $p \mapsto (Xf)(p) := X_p f$. (Do not confuse the notations fX and Xf : the former is a smooth vector field on U obtained by multiplying X by f, while the latter is the real-valued fnct on U obtained by applying the vector field X to the smooth fnct f.) Since the action of a tangent vector on a fnct is determined by the values of the fnct in an arbitrarily small neighborhood (PROP. 3.8), it follows that Xf is locally determined. In particular, for any open subset $V \subseteq U$, we have $(Xf)|_V = X(f|_V)$.

This construction yields another useful smoothness criterion for vector fields (see also PROP. 6.13 for another one).

· PROP. 7.5 (Smoothness criterion for vector fields): Let M be a smooth mnfd and let $X: M \longrightarrow TM$ be a rough vector field. T.f.a.e.:

(a) X is smooth.

(b) $\forall f \in C^\infty(M)$: $Xf : M \longrightarrow \mathbb{R}$ is smooth.

(c) $\forall U \subseteq M$ open $\forall f \in C^\infty(U)$: $Xf : U \longrightarrow \mathbb{R}$ is smooth.

PROOF:

(a) $\Rightarrow$ (b): Given $p \in M$, take a smooth chart $(U,(x^i))$ for M containing p. For $x \in U$ we may write

$$Xf(x) = \left(X^i(x) \frac{\partial}{\partial x^i}\Big|_x \right) f = X^i(x) \frac{\partial \hat{f}}{\partial x^i}(\hat{x}).$$

Since the component fncts X^i of X are smooth on U by PROP. 7.3, it follows that Xf is smooth on U. We conclude by ES3E2(a).

(b) $\Rightarrow$ (c): Fix $U \subseteq M$ open and $f \in C^\infty(U)$. For any $p \in U$, let ψ be a smooth bump fnct that is equal to 1 in a neighborhood of p and supported in U (see PROP. 2.14), and define $\tilde{f} = \psi f$, extended to be zero on $M \setminus \operatorname{supp}\psi$. Then $X\tilde{f}$ is smooth by assumption, and is equal to Xf in a neighborhood of p (by the discussion on p. 92). We conclude by ES3E2(a).

(c) $\Rightarrow$ (a): If (x^i) are smooth local coordinates on $U \subseteq M$, then we can think of each coordinate x^i as a smooth fnct on U, and we have

$$X(x^i) = \left(X^j \frac{\partial}{\partial x^j}\right)(x^i) \xlongequal{\partial x^i/\partial x^j = \delta_{ij}} X^i,$$

which is smooth by assumption. We conclude by PROP. 7.3. ∎

One consequence of PROP. 7.5 is that a smooth vector field $X \in \mathfrak{X}(M)$ defines a map

$$C^\infty(M) \longrightarrow C^\infty(M),\ f \mapsto Xf,$$

which is $\mathbb{R}$-linear and satisfies the following product rule for vector fields:

$$X(fg) = f\,Xg + g\,Xf.$$

(check this pointwise); in other words, this map is a <u>derivation of $C^\infty(M)$</u>.

The next proposition shows that derivations of $C^\infty(M)$ can be identified with smooth vector fields (and thus we sometimes use the same letter for both the smooth vector field (thought of as a map $M \to TM$) and the derivation of $C^\infty(M)$).

PROP. 7.6: Let M be a smooth mnfd. A map $D: C^\infty(M) \to C^\infty(M)$ is a derivation iff it is of the form $Df = Xf$ for some $X \in \mathfrak{X}(M)$.

PROOF:

"$\Leftarrow$": We just showed above that any smooth vector field induces a derivation of $C^\infty(M)$.

"$\Rightarrow$": Let $p \in M$ and consider the map

$$X_p : C^\infty(M) \to \mathbb{R}, \ f \mapsto (Df)(p).$$

Since D is $\mathbb{R}$-linear, X_p is also $\mathbb{R}$-linear, and since D is a derivation, we have

$$\begin{aligned} X_p(fg) &= D(fg)(p) = \big(fD(g) + gD(f)\big)(p) \\ &= f(p)\,D(g)(p) + g(p)\,D(f)(p) \\ &= f(p)\,X_p g + g(p)\,X_p f. \end{aligned}$$

Hence, X_p is a derivation at $p \in M$, i.e., $X_p \in T_pM$. We obtain thus a rough vector field $X: M \to TM$, $p \mapsto X_p$, but since $Xf = Df$ is smooth for every $f \in C^\infty(M)$, X is actually smooth by PROP 7.5, and we are done. ∎

Utilizing PROP. 7.6, we now introduce an important way of combining two smooth vector fields to obtain another smooth vector field.

Let M be a smooth mnfd and let $X, Y \in \mathfrak{X}(M)$. Given $f \in C^\infty(M)$, we can apply X to f to obtain $Xf \in C^\infty(M)$ (see PROP. 7.5) and we can now apply Y to Xf to obtain $Y(Xf) \in C^\infty(M)$. The operation $f \mapsto YXf$, though, does not satisfy the product rule in general, and thus cannot be a vector field (see PROP. 7.6), as the following example shows:

· EXAMPLE 7.7: Consider the vector fields

$$X = \frac{\partial}{\partial x} \quad \text{and} \quad Y = x\frac{\partial}{\partial y}$$

and the smooth fncts

$$f(x,y) = x \quad \text{and} \quad g(x,y) = y$$

on $\mathbb{R}^2$. We compute

$$XY(fg) = X\left(x\frac{\partial(xy)}{\partial y}\right) = X(x^2) = \frac{\partial x^2}{\partial x} = 2x,$$

$$fXYg + gXYf = xX\left(x\cancelto{1}{\frac{\partial y}{\partial y}}\right) + yX\left(x\cancelto{0}{\frac{\partial x}{\partial y}}\right) = x\frac{\partial x}{\partial x} = x,$$

so XY is not a derivation of $C^\infty(\mathbb{R}^2)$.

We can also apply the same two vector fields in the opposite order, obtaining a (usually different) smooth fnct YXf. Applying both these operators to $f \in C^\infty(M)$ and subtracting, we obtain an operator

$$[X,Y]: C^\infty(M) \longrightarrow C^\infty(M), \quad f \mapsto XYf - YXf,$$

called <u>the Lie bracket of X and Y</u>. We will now show that $[X,Y]$ is a derivation of $C^\infty(M)$, and hence $[X,Y] \in \mathfrak{X}(M)$ by PROP. 7.6.

· $\mathbb{R}$-linearity: $[X,Y](af+bg) = a[X,Y]f + b[X,Y]g$
(follows from the $\mathbb{R}$-linearity of X and Y)

· Product rule: $[X,Y](fg) = XY(fg) - YX(fg) =$

$$= X(fYg + gYf) - Y(fXg + gXf)$$

$$= [\cancel{Xf\,Yg} + fXYg + \cancel{Xg\,Yf} + gXYf] -$$

$$[\cancel{Yf\,Xg} + fYXg + \cancel{Yg\,Xf} + gYXf]$$

$$= f(XYg - YXg) + g(XYf - YXf)$$

$$= f[X,Y]g + g[X,Y]f.$$

$\rightsquigarrow$ for a geometric interpretation of the Lie bracket, see [Lee, § 9.4 "Lie derivatives"]

$\rightsquigarrow$ for basic properties of the Lie bracket, see ES11 and ES12E1.

If $S \subseteq M$ is an immersed or embedded submnfd, a vector field X on M does not necessarily restrict to a vector field on S, because $X_p \in T_pM$ may not lie in the subspace $T_pS \subseteq T_pM$ at a pt $p \in S$. Given a pt $p \in S$, a vector field X on M is said to be <u>tangent to S at p</u> if $X_p \in T_pS \subseteq T_pM$, and <u>tangent to S</u> if it is tangent to S at all pts of S.

The following result is an immediate consequence of PROP. 5.17:

· <u>PROP. 7.8</u>: Let M be a smooth mnfd, $S \subseteq M$ be an embedded submnfd, and $X \in \mathfrak{X}(M)$. Then X is tangent to S iff $(Xf)|_S = 0$ for every $f \in C^\infty(M)$ s.t. $f|_S \equiv 0$.

Let M be a smooth mnfd. If $\gamma : J \subseteq \mathbb{R} \longrightarrow M$ is a smooth curve, then for each $t \in J$ the velocity vector $\gamma'(t)$ is an element of $T_{\gamma(t)}M$. We describe next a way to work backwards: given a tangent vector at each pt, we seek a curve whose velocity at each pt is equal to the given vector there.

<u>DEF. 7.9</u>: Let M be a smooth mnfd and let V be a vector field on M. An <u>integral curve of V</u> is a differentiable curve $\gamma : J \longrightarrow M$ whose velocity at each pt is equal to the value of V at that pt:

$$\gamma'(t) = V_{\gamma(t)} \quad , \forall t \in J .$$

If $0 \in J$, then $\gamma(0) \in M$ is called <u>the starting pt of γ</u>.

Finding integral curves of vector fields boils down to solving a system of ODEs in a smooth chart: Suppose that $V \in \mathfrak{X}(M)$ and that $\gamma : J \to M$ is a smooth curve. On a smooth coordinate domain $U \subseteq M$ we can write γ in local coordinates as $\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t))$. Then the condition $\gamma'(t) = V_{\gamma(t)}$ for γ to be an integral curve of V can be written

$$\dot{\gamma}^i(t) \left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)} = V^i(\gamma(t)) \left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)} ,$$

which reduces to the following autonomous system of ODEs:

$$\begin{cases} \dot{\gamma}^1(t) = V^1(\gamma^1(t), \dots, \gamma^n(t)) \\ \vdots \\ \dot{\gamma}^n(t) = V^n(\gamma^1(t), \dots, \gamma^n(t)) \end{cases}$$

The fundamental fact about such systems is the following existence, uniqueness and smoothness thm:

· <u>THM</u>: Let $V: U \to \mathbb{R}^n$ be a smooth vector-valued fnct, where $U \subseteq \mathbb{R}^n$ is open. Consider the initial value problem

$$\dot{y}^i(t) = V^i(y^1(t), \dots, y^n(t)) \quad , \quad 1 \leq i \leq n \qquad (1)$$

$$y^i(t_0) = c^i \quad , \quad 1 \leq i \leq n \qquad (2)$$

for arbitrary $t_0 \in \mathbb{R}$ and $c = (c^1, \dots, c^n) \in \mathbb{R}^n$.

(a) Existence: For any $t_0 \in \mathbb{R}$ and $x_0 \in U$, there exists an open interval $J_0 \ni t_0$ and an open subset $x_0 \in U_0 \subseteq U$ s.t. for each $c \in U_0$ there is a C^1 map $y: J_0 \to U$ that solves (1)-(2).

(b) Uniqueness: Any two differentiable solutions to (1)-(2) defined on intervals containing t_0 agree on their common domain.

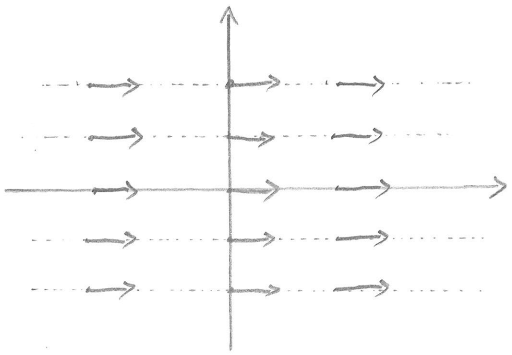
(c) Smoothness: Let J_0 and U_0 be as in (a), and consider the map $\theta: J_0 \times U_0 \to U$, $(t, x) \mapsto y(t)$, where $y: J_0 \to U$ is the unique solution to (1) with initial condition $y(t_0) = x$. Then θ is smooth.

An easy consequence of this thm is the following result.

· <u>PROP. 7.10</u>: Let V be a smooth vector field on a smooth mnfd M. For each pt $p \in M$, there exists $\varepsilon > 0$ and a (unique) smooth curve $\gamma: (-\varepsilon, \varepsilon) \to M$ that is an integral curve of V starting at $p \in M$.

EXAMPLE 7.11: Let (x,y) be the standard coordinates on $\mathbb{R}^2$.

1) Consider $V = \frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^2)$. The integral curves of V are precisely the straight lines parallel to the x-axis, with parametrizations of the form $\gamma(t) = (a+t, b)$ for constants $a, b \in \mathbb{R}$.



Thus, there is a unique integral curve starting at each pt of the plane, and the images of different integral curves are either identical or disjoint. $\uparrow$ L11 $\downarrow$ L12

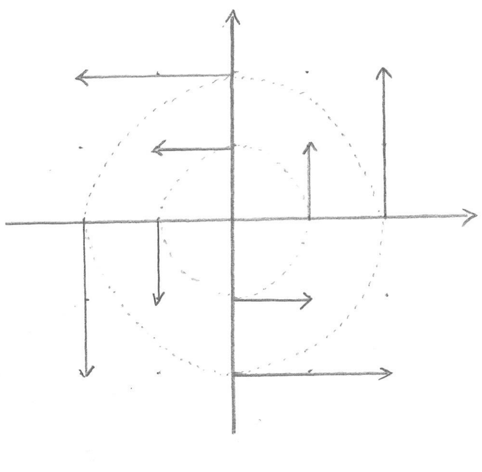
2) Consider $W = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2)$. To determine the integral curves of W we proceed as follows (see p. 97):

$$\gamma(t) = (\gamma^1(t), \gamma^2(t)) \rightsquigarrow \dot{\gamma}(t) = W_{\gamma(t)} \Rightarrow$$

$$\Rightarrow \begin{cases} \dot{\gamma}_1(t) = -\gamma_2(t) \\ \dot{\gamma}_2(t) = \gamma_1(t) \end{cases} \xrightarrow{\ddot{\gamma}_1(t) + \gamma_1(t) = 0}$$

$$\Rightarrow \begin{cases} \gamma_1(t) = a\cos t - b\sin t \\ \gamma_2(t) = a\sin t + b\cos t \quad (= -\dot{\gamma}_1(t)) \end{cases}$$

for constants $a, b \in \mathbb{R}$. Thus, each curve of the form



$$\gamma(t) = (a\cos t - b\sin t, a\sin t + b\cos t), \ t \in \mathbb{R}$$

is an integral curve of W. When $(a,b) = (0,0)$, this is the constant curve $\gamma(t) \equiv (0,0)$, otherwise, it is a circle traversed clockwise. Since $\gamma(0) = (a,b)$, we see again that there is a unique integral curve starting at