Midterm Solutions

Exercice 1. (21 + 3 points)

On the state space $S = \{0, 1\}^d$, with $d \geq 2$ integer, consider the time-homogeneous Markov chain $(X_n, n \geq 0)$ with the following behaviour:

At each time step $n \geq 0$, choose a number $i \in \{1, \ldots, d\}$ uniformly at random, and also independently of the numbers chosen at previous time steps; then assuming that $X_n = x$, set

$$(X_{n+1})_i = \begin{cases} 0 & \text{with probability } \frac{|x|}{d} \\ 1 & \text{with probability } 1 - \frac{|x|}{d} \end{cases}$$

as well as $(X_{n+1})_j = x_j$ for every $j \neq i$.

NB: Recall that $|x|$ stands for the number of 1’s in the vector $x$.

a) Explain why the process $(X_n, n \geq 0)$ is a Markov chain (a short justification will do here).

**Answer:** In short, the steps performed by the process are independent from each other, so the Markov property holds. In more detail, we have

$$\mathbb{P}(X_{n+1} = y | X_n = x, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) =$$

$$\begin{cases} \frac{1}{d} \left(1 - \frac{|x|}{d}\right) y_i \left(\frac{|x|}{d}\right)^{1-y_i} & \text{if } x \text{ and } y \text{ differ only at the } i^{th} \text{ coordinate} \\ \frac{1}{d} \sum_{i=1}^d \left(1 - \frac{|x|}{d}\right) y_i \left(\frac{|x|}{d}\right)^{1-y_i} & \text{if } y = x \\ 0 & \text{otherwise}, \end{cases}$$

which depends only on $x$ and $y$, and hence $(X_n, n \geq 0)$ is a Markov chain.

b) Do the states $x_0 = (0, 0, \ldots, 0)$ and $x_1 = (1, 1, \ldots, 1)$ communicate in this chain? Is the chain irreducible? Justify.

**Answer:** Yes, $x_0$ and $x_1$ communicate. The Markov chain is irreducible since any two states communicate with each other.

c) Is the chain aperiodic for any value of $d \geq 2$? Justify.

**Answer:** The chain is aperiodic for $d \geq 2$. This is since all the states except $x_0 = (0, 0, \ldots, 0)$ and $x_1 = (1, 1, \ldots, 1)$ have self-loops.

*From now on, we restrict ourselves to the particular case $d = 3$.*

d) Write down explicitly the transition matrix $P$ of the chain (it might perhaps be a good idea to also draw the transition graph, but this is not required here).
Answer: Considering the order (000), (001), (010), (011), (100), (101), (110), (111) for the states of \( S \), we obtain

\[ P = \begin{pmatrix}
0 & 1/3 & 1/3 & 0 & 1/3 & 0 & 0 & 0 \\
1/9 & 4/9 & 0 & 2/9 & 0 & 2/9 & 0 & 0 \\
1/9 & 0 & 4/9 & 2/9 & 0 & 0 & 2/9 & 0 \\
0 & 2/9 & 2/9 & 4/9 & 0 & 0 & 0 & 1/9 \\
1/9 & 0 & 0 & 0 & 4/9 & 2/9 & 2/9 & 0 \\
0 & 2/9 & 0 & 0 & 2/9 & 4/9 & 0 & 1/9 \\
0 & 0 & 2/9 & 0 & 2/9 & 0 & 4/9 & 1/9 \\
0 & 0 & 0 & 1/3 & 0 & 1/3 & 1/3 & 0
\end{pmatrix} \]

e) Compute the (unique) stationary distribution \( \pi \) of the chain. Does detailed balance hold?

Answer: Let’s see if there exists a \( \pi \) that satisfies the detailed balance equation. We have

\[ \pi_2 = \frac{p_{12}}{p_{21}} \pi_1 \implies \pi_2 = 3 \pi_1. \]

Similarly, we get

\[ \begin{align*}
\pi_3 &= 3 \pi_1, \\
\pi_4 &= \pi_2 = 3 \pi_1, \\
\pi_5 &= 3 \pi_1, \\
\pi_6 &= \pi_2 = 3 \pi_1, \\
\pi_7 &= \pi_3 = 3 \pi_1, \\
\pi_8 &= \pi_4/3 = \pi_1.
\end{align*} \]

Imposing \( \sum \pi_i = 1 \), we get \( \pi_1 = \frac{1}{20} \). Hence, \( \pi = \frac{1}{20} (1, 3, 3, 3, 3, 3, 3, 1) \) and the chain satisfies detailed balance.

f) Is \( \pi \) also a limiting distribution? Justify.

Answer: The given Markov chain is irreducible, positive-recurrent (due to finite state space), and aperiodic, therefore ergodic. Hence \( \pi \) is also the limiting distribution.

BONUS g) Explain what major change(s) would happen in the Markov chain \((X_n, n \geq 0)\) if we inverted the roles of 0 and 1 in equation (1).

Answer: There would be 3 equivalence classes: \( \{x_0 = (0, 0, \ldots, 0)\} \), and \( \{x_1 = (1, 1, \ldots, 1)\} \) that are recurrent and \( \{0, 1\}^d \setminus \{x_0, x_1\} \) which is transient.
Exercice 2. (17 points)

Let \((X_n, n \geq 0)\) be a Markov chain with state space \(S = \{0, 1, 2, 3\}\) and transition matrix \(P\) given by

\[
P = \begin{pmatrix}
0 & 1 & 0 & 0 \\
 a & b & b & 0 \\
0 & b & b & a \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

where the parameters \(0 \leq a \leq 1\) and \(0 \leq b \leq \frac{1}{2}\) are such that \(a + 2b = 1\).

a) For what values of \(a\) is the chain \((X_n, n \geq 0)\) ergodic? Justify.

Answer: The chain is ergodic for \(a \in (0, 1)\). For \(a \in \{0, 1\}\), the chain is reducible.

b) Compute the unique limiting and stationary distribution \(\pi\) of the chain in this case.

Answer: Solving \(\pi = \pi P\) and using \(a + 2b = 1\), we get \(\pi = \frac{1}{2(1+a)} (a, 1, 1, a)\).

c) Compute the eigenvalues \(\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3\) of the matrix \(P\).

Hint: Look for eigenvectors of the form \((u, v, v, u)\) and \((u, v, -v, -u)\).

Answer: We have

\[
P \begin{pmatrix} u \\ v \\ v \\ u \end{pmatrix} = \begin{pmatrix} v \\ au + 2bv \\ au + 2bv \\ v \end{pmatrix} = \begin{pmatrix} u \\ u(u/v + 2b) \\ u(u/v + 2b) \\ u \end{pmatrix}.
\]

Hence \((u, v, v, u)^T\) is an eigenvector for \(u, v\) such that \(\frac{u}{v}(a\frac{v}{u} + 2b) = 1\) which is satisfied for \(u = v\) and \(v = -au\). Corresponding eigenvalues are 1 and \(-a\). Also, we have

\[
P \begin{pmatrix} u \\ v \\ -v \\ -u \end{pmatrix} = \begin{pmatrix} v \\ au \\ -au \\ -v \end{pmatrix} = \begin{pmatrix} u \\ u(a^2/v) \\ -u(a^2/v) \\ -u \end{pmatrix}.
\]

Hence \([u, v, -v, -u]^T\) is an eigenvector for \(u, v\) such that \(\frac{au^2}{v} = v\) which is satisfied for \(v = \pm \sqrt{au}\). Corresponding eigenvalues are \(\pm \sqrt{a}\). Therefore the eigenvalues of \(P\) are \(1 \geq \sqrt{a} \geq -a \geq -\sqrt{a}\).

d) Express the spectral gap \(\gamma\) of the chain as a function of the parameter \(a\).

Answer: Since \(a \in [0, 1]\), we have \(\lambda_* = \max\{\lambda_1, -\lambda_3\} = \sqrt{a}\). Hence the spectral gap \(\gamma = 1 - \sqrt{a}\).

e) Is there a way to increase the spectral gap \(\gamma\) by making the chain lazy (i.e., adding self-loops of equal weight to each state)? Justify.

Answer: Let us add self loops of strength \(\alpha \in (0, 1)\) to each state. Then, the new transition probability matrix is \(\alpha I + (1 - \alpha)P\) whose eigenvalues are 1, \(\alpha + (1 - \alpha)\sqrt{a}\), \(\alpha + (1 - \alpha)a\), \(\alpha - (1 - \alpha)\sqrt{a}\). The eigenvalue \(\alpha + (1 - \alpha)\sqrt{a}\) is between \(\sqrt{a}\) and 1 for any \(\alpha\). Hence we can only reduce the spectral gap.
Exercise 3. (12 points)

Let \((X_n, n \in \mathbb{N})\) be a sequence of i.i.d. random variables such that \(\mathbb{P}(X_n = 0) = \mathbb{P}(X_n = 1) = \mathbb{P}(X_n = 2) = \frac{1}{3}\) for every \(n \geq 0\).

a) Let \((Y_n, n \in \mathbb{N})\) be the process defined as

\[ Y_n = \prod_{j=0}^{n} X_j, \quad n \geq 0 \]

Explain why \((Y_n, n \in \mathbb{N})\) is a Markov chain. Is it time-homogeneous?

**Answer:** We have

\[
\mathbb{P}(Y_{n+1} = j|Y_n = i, Y_{n-1} = y_{n-1}, \ldots, Y_0 = y_0) = \mathbb{P}\left(\prod_{l=0}^{n+1} X_l = j \bigg| \prod_{l=0}^{n} X_l = i, Y_{n-1} = y_{n-1}, \ldots, Y_0 = y_0\right)
\]

\[= \mathbb{P}\left( X_{n+1} = j/i \bigg| \prod_{l=0}^{n} X_l = i, Y_{n-1} = y_{n-1}, \ldots, Y_0 = y_0\right) = \mathbb{P}(X_{n+1} = j/i) \]

Therefore \((Y_n, n \in \mathbb{N})\) is a time-homogeneous Markov chain.

b) What is the state space \(S\) of the chain \((Y_n, n \in \mathbb{N})\)? And which states are recurrent/transient?

**Answer:** The state space of \((Y_n, n \in \mathbb{N})\) is \(S = \{2^n : n \in \mathbb{N}\} \cup \{0\}\). State 0 is recurrent. All other states are transient.

c) Let now \((Z_n, n \in \mathbb{N})\) be the process defined as

\[ Z_n = \max_{0 \leq j \leq n} Y_j, \quad n \geq 0 \]

Is \((Z_n, n \in \mathbb{N})\) also a Markov chain? Is it time-homogeneous? Justify.

**Answer:** \((Z_n, n \in \mathbb{N})\) is not a Markov chain. Consider the sample paths that can make \(Z_1 = 2\). They are \((X_0, X_1) \in \{(1, 2), (2, 1), (2, 0)\}\). The corresponding values of \((Z_0, Z_1)\) are \((1, 2), (2, 2), (2, 2)\). Hence we have

\[ \mathbb{P}(Z_2 = 2|Z_1 = 2) = \frac{7}{9} \quad \text{and} \quad \mathbb{P}(Z_2 = 4|Z_1 = 2) = \frac{2}{9}, \]

while

\[ \mathbb{P}(Z_2 = 2|Z_1 = 2, Z_0 = 1) = \frac{2}{3} \quad \text{and} \quad \mathbb{P}(Z_2 = 4|Z_1 = 2, Z_0 = 1) = \frac{1}{3}. \]

Therefore, \((Z_n, n \in \mathbb{N})\) is not a Markov chain (and therefore neither time-homogeneous).