**Exercice 1. (21 + 3 points)**

On the state space \( S = \{0, 1\}^d \), with \( d \geq 2 \) integer, consider the time-homogeneous Markov chain \((X_n, n \geq 0)\) with the following behaviour:

At each time step \( n \geq 0 \), choose a number \( i \in \{1, \ldots, d\} \) uniformly at random, and also independently of the numbers chosen at previous time steps; then assuming that \( X_n = x \), set

\[
(X_{n+1})_i = \begin{cases} 
0 & \text{with probability } \frac{|x|}{d} \\
1 & \text{with probability } 1 - \frac{|x|}{d}
\end{cases}
\]

as well as \((X_{n+1})_j = x_j\) for every \( j \neq i\).

**NB:** Recall that \(|x|\) stands for the number of 1’s in the vector \(x\).

a) Explain why the process \((X_n, n \geq 0)\) is a Markov chain (a short justification will do here).

b) Do the states \(x_0 = (0, 0, \ldots, 0)\) and \(x_1 = (1, 1, \ldots, 1)\) communicate in this chain? Is the chain irreducible? Justify.

c) Is the chain aperiodic for any value of \(d \geq 2\)? Justify.

*From now on, we restrict ourselves to the particular case \(d = 3\).*

d) Write down explicitly the transition matrix \(P\) of the chain (it might perhaps be a good idea to also draw the transition graph, but this is not required here).

e) Compute the (unique) stationary distribution \(\pi\) of the chain. Does detailed balance hold?

f) Is \(\pi\) also a limiting distribution? Justify.

**BONUS g)** Explain what major change(s) would happen in the Markov chain \((X_n, n \geq 0)\) if we inverted the roles of 0 and 1 in equation (1).
Exercice 2. (17 points)

Let \((X_n, n \geq 0)\) be a Markov chain with state space \(S = \{0, 1, 2, 3\}\) and transition matrix \(P\) given by

\[
P = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & b & b & 0 \\
0 & b & b & a \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

where the parameters \(0 \leq a \leq 1\) and \(0 \leq b \leq \frac{1}{2}\) are such that \(a + 2b = 1\).

a) For what values of \(a\) is the chain \((X_n, n \geq 0)\) ergodic? Justify.

b) Compute the unique limiting and stationary distribution \(\pi\) of the chain in this case.

c) Compute the eigenvalues \(\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3\) of the matrix \(P\).

*Hint:* Look for eigenvectors of the form \((u, v, v, u)\) and \((u, v, -v, -u)\).

d) Express the spectral gap \(\gamma\) of the chain as a function of the parameter \(a\).

e) Is there a way to increase the spectral gap \(\gamma\) by making the chain lazy (i.e., adding self-loops of equal weight to each state)? Justify.

Exercice 3. (12 points)

Let \((X_n, n \in \mathbb{N})\) be a sequence of i.i.d. random variables such that \(\mathbb{P}(X_n = 0) = \mathbb{P}(X_n = 1) = \mathbb{P}(X_n = 2) = \frac{1}{3}\) for every \(n \geq 0\).

a) Let \((Y_n, n \in \mathbb{N})\) be the process defined as

\[
Y_n = \prod_{j=0}^{n} X_j, \quad n \geq 0
\]

Explain why \((Y_n, n \in \mathbb{N})\) is a Markov chain. Is it time-homogeneous?

b) What is the state space \(S\) of the chain \((Y_n, n \in \mathbb{N})\)? And which states are recurrent/transient?

c) Let now \((Z_n, n \in \mathbb{N})\) be the process defined as

\[
Z_n = \max_{0 \leq j \leq n} Y_j, \quad n \geq 0
\]

Is \((Z_n, n \in \mathbb{N})\) also a Markov chain? Is it time-homogeneous? Justify.