# Final Exam - "Signals" Part

January 19, 2022

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## Exercise 1

Given the following time domain signal:

$$x(t) = \left[1 + \text{sinc}(t)e^{j2\pi f_0 t}\right] \sin(\pi f_0 t) \cos(\pi f_0 t)$$

Compute its Fourier transform $X(f) = \mathcal{F}\{x(t)\}$.
*Hint: Use the properties of the Fourier transform.*

## Solution 1

We start by observing that:

$$\sin(\pi f_0 t) \cos(\pi f_0 t) = \frac{1}{2} \sin(2\pi f_0 t)$$

Using the fact that a multiplication in the time domain is a convolution in the frequency domain, and the linearity of the transform we obtain:

$$\begin{aligned}
\mathcal{F}\{x(t)\} &= \mathcal{F}\{\frac{1}{2}\left[1 + \text{sinc}(t)e^{j2\pi f_0 t}\right] \sin(2\pi f_0 t)\} \\
&= \mathcal{F}\{\frac{1}{2}\left[1 + \text{sinc}(t)e^{j2\pi f_0 t}\right]\} * \mathcal{F}\{\sin(2\pi f_0 t)\} \\
&= \frac{1}{2}\mathcal{F}\{\left[1 + \text{sinc}(t)e^{j2\pi f_0 t}\right]\} * \mathcal{F}\{\sin(2\pi f_0 t)\} \\
&= \frac{1}{2}\left[\mathcal{F}\{1\} + \mathcal{F}\{\text{sinc}(t)e^{j2\pi f_0 t}\}\right] * \mathcal{F}\{\sin(2\pi f_0 t)\}
\end{aligned}$$

Using the following properties:

1. $\mathcal{F}\{1\} = \delta(f)$.
2. $\mathcal{F}\{\sin(2\pi f_0 t)\} = \frac{i}{2}[\delta(f + f_0) - \delta(f - f_0)]$.
3. $\mathcal{F}\{x(t)e^{j2\pi f_0 t}\} = X(f - f_0)$, when $\mathcal{F}\{x(t)\} = X(f)$.
4. $\mathcal{F}\{\text{sinc}(t)\} = \text{rect}(f)$.

Hence:

$$\mathcal{F}\{x(t)\} = \frac{1}{2}\left[\delta(f) + \text{rect}(f - f_0)\right] * \frac{i}{2}[\delta(f + f_0) - \delta(f - f_0)]$$

From the definition of the convolution we have:

$$\begin{aligned}
\mathcal{F}\{x(t)\} &= \frac{i}{4}\int_{-\infty}^{\infty}[\delta(\tau)+\text{rect}(\tau-f_0)][\delta(f+f_0-\tau)-\delta(f-f_0-\tau)]d\tau \\
&= \frac{i}{4}\Bigg[\int_{-\infty}^{\infty}\delta(\tau)\delta(f+f_0-\tau)d\tau - \int_{-\infty}^{\infty}\delta(\tau)\delta(f-f_0-\tau)d\tau \\
&+ \int_{-\infty}^{\infty}\delta(f+f_0-\tau)\,\text{rect}(\tau-f_0)d\tau - \int_{-\infty}^{\infty}\delta(f-f_0-\tau)\,\text{rect}(\tau-f_0)d\tau\Bigg] \\
&= \frac{i}{4}[\delta(f+f_0)-\delta(f-f_0)+\text{rect}(f+f_0-f_0)-\text{rect}(f-f_0-f_0)] \\
&= \frac{i}{4}[\delta(f+f_0)-\delta(f-f_0)+\text{rect}(f)-\text{rect}(f-2f_0)]
\end{aligned}$$

Therefore, we have:

$$X(f) = \frac{i}{4}[\delta(f+f_0)-\delta(f-f_0)+\text{rect}(f)-\text{rect}(f-2f_0)]$$

# Exercise 2

Given the following continuous, time domain signal:

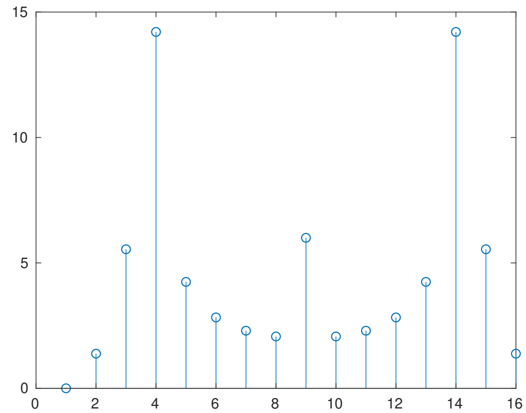
$$x(t) = 2\sin(20\pi t + \frac{\pi}{2}) + \cos(60\pi t + \frac{2\pi}{3}).$$

1. Compute its continuous Fourier transform $X(f) = \mathcal{F}\{x(t)\}$.

2. Suppose now that the signal is first sampled at $f_s = 60$Hz, give the expression of the hereby obtained discrete signal $x[n] = x(nT_s)$, where $T_s = 1/f_s$.

3. Suppose that the DFT is computed over the first $N = 12$ samples of the discrete signal ($n = 0, ..., 11$), provide the indices of the non-empty frequency bins, and their corresponding continuous frequency. *Hint: you do not need to actually compute the DFT to answer this question.*

4. The DFT is now performed over $N = 16$ points and we obtained the following result:




Did you expect this result? Explain what caused the difference compared to the case $N = 12$.

# Solution 2

1. Using the following properties:

   (a) $\mathcal{F}\{\sin(2\pi f_0 t)\} = \frac{i}{2}[\delta(f + f_0) - \delta(f - f_0)]$.

   (b) $\mathcal{F}\{\cos(2\pi f_0 t)\} = \frac{1}{2}[\delta(f + f_0) + \delta(f - f_0)]$.

   (c) $\mathcal{F}\{x(t - t_0)\} = X(f)e^{-j2\pi f t_0}$, when $\mathcal{F}\{x(t)\} = X(f)$.

   We obtain:

$$X(f) = [\delta(f + 10) + \delta(f - 10)] + \frac{1}{2}\left[e^{j\frac{2\pi}{3}}\delta(f - 30) + e^{-j\frac{2\pi}{3}}\delta(f + 30)\right]$$

2. The signal sampled at $f_s = 60$Hz is given by:

$$\begin{aligned}
x[n] &= x(nT_s) && (1)\\
&= 2\sin\left(20\pi n T_s + \frac{\pi}{2}\right) + \cos\left(60\pi n T_s + \frac{2\pi}{3}\right) && (2)\\
&= 2\cos\left(\frac{20\pi n}{f_s}\right) + \cos\left(\frac{60\pi n}{f_s} + \frac{2\pi}{3}\right) && (3)\\
&= 2\cos(\frac{\pi}{3}n) + \cos(\pi n + \frac{2\pi}{3}) && (4)\\
&= 2\cos(\frac{\pi}{3}n) - \frac{1}{2}(-1)^n && (5)
\end{aligned}$$

3. By directly applying the DFT on the discrete signal we get:

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n] e^{-j2\pi\frac{kn}{N}} \\
&= \sum_{n=0}^{11} e^{j(\frac{\pi}{3}-\frac{k\pi}{6})n} + \sum_{n=0}^{11} e^{j(-\frac{\pi}{3}-\frac{k\pi}{6})n} + \frac{e^{j\frac{2\pi}{3}}}{2} \sum_{n=0}^{11} e^{j(\pi-\frac{k\pi}{6})n} + \frac{e^{-j\frac{2\pi}{3}}}{2} \sum_{n=0}^{11} e^{j(-\pi-\frac{k\pi}{6})n}
\end{aligned}
$$

We observe that we have a sum of geometric series, and use $\sum_{n=0}^{N-1} q^n = \frac{1-q^N}{1-q}$:

$$
\sum_{n=0}^{11} e^{j(2\pi\frac{l}{12}-\frac{k\pi}{6})n} = \frac{1-e^{j(2\pi l - 2\pi k)}}{1-e^{j(2\pi\frac{l}{12}-\frac{k\pi}{6})}} = \begin{cases} 12 \text{ if } k = 12m + l,\ m, l \in \mathbb{Z} \\ 0 \text{ otherwise} \end{cases}
$$

Therefore:

$$
X[k] = 12\delta(k-2) \ + \ 12\delta(k+2) \ + \ 6\delta(k-6), \quad \text{with } k \in [-5, 6] \cap \mathbb{Z}.
$$

This result is coherent with that of the continuous case. We indeed have well defined peaks at the frequencies corresponding to ±10Hz and ±30Hz. As 30Hz is precisely $f_s/2$ the positive and negative peaks are situated at the same discrete position.

4. This result is expected. Indeed, the continuous signal is composed of two frequencies, namely $f_1 =$ 10Hz and $f_2 = 30$Hz, with a sampling frequency $f_s = 60$Hz and $N = 12$ both of them are part of the harmonics. On the other hand, when the number of points is set to $N = 16$ the frequency resolution is $\Delta f = 3.75$Hz and $f_1 = 10$Hz is not part of the harmonics anymore, hence the observed result.