Not all embedded submnfds can be expressed as level sets of smooth submersions. However, the next proposition shows that every embedded submnfd is at least locally of this form.

· PROP. 5.11 : Let S be a subset of a smooth m-mnfd M. Then S is an embedded k-submnfd of M iff every pt of S has a neighborhood U in M s.t. $U \cap S$ is a level set of a smooth submersion $\Phi : U \to \mathbb{R}^{m-k}$.

PROOF : ES8E4. ∎

If $S \subseteq M$ is an embedded submnfd, a smooth map $\Phi : M \to N$ s.t. S is a regular level set of Φ is called a defining map for S. (In the special case $N = \mathbb{R}^{m-k}$ it is usually called a defining function for S. For several examples, see ES8 and ES9.). More generally, if $U \subseteq M$ is an open subset and $\Phi : U \to N$ is a smooth map s.t. $S \cap U$ is a regular level set of Φ, then Φ is called a local defining map (or local defining fnct) for S. PROP. 5.11 says that every embedded submnfd admits a local defining fnct in a neighborhood of each of its pts.

· DEF. 5.12 : Let M be a smooth mnfd. An immersed submanifold of M is a subset $S \subseteq M$ endowed with a topology (not necessarily the subspace top.) w.r.t. which it is a top. mnfd, and a smooth structure w.r.t. which the inclusion map $S \hookrightarrow M$ is (an injective) smooth immersion. The codimension of S in M is defined as $\dim M - \dim S$.

Observe that every embedded submnfd is an immersed submnfd, but the converse fails in general; see, for instance, ES8E3(b) and ES9E4(b) for a counterexample.

·<u>PROP. 5.13</u> (Images of immersions as submanifolds): Let $F: N \to M$ be an injective smooth immersion. Set $S := F(N)$. Then S has a unique topology and smooth structure s.t. it is an immersed submnfd of M and s.t. $F: N \to S$ is a diffeomorphism onto its image.

<u>PROOF</u>: We give S a topology by declaring a subset $U \subseteq S$ to be open iff $F^{-1}(U) \subseteq N$ is open, and then we give it a smooth structure by taking the smooth charts to be those of the form $(F(U), \varphi \circ F^{-1})$, where (U, φ) is a smooth chart for N. (As in the proof of PROP. 5.3, the smooth compatibility of these charts follows from the smooth compatibility of the corresponding charts for N.) With this top. and smooth structure on S, the map F is a diffeomorphism onto its image, and these are the only top. and smooth structure on S with this property. The inclusion map $\iota: S \hookrightarrow M$ can be written as the composition

$$S \xrightarrow{F^{-1}} N \xrightarrow{F} M,$$

where the first map is a diffeomorphism and the second map is a smooth immersion, so ι is also a smooth immersion by ES6E1(a)(ii) and ES6E5(a). ■

· **EXAMPLE 5.14**: The figure-eight (lemniscate) from EX. 4.5(2) is the image of the injective smooth immersion
$$\beta : (-\pi, \pi) \longrightarrow \mathbb{R}^2,\ t \mapsto (\sin 2t, \sin t)$$
(which is not an embedding), so it is an immersed submnfd of $\mathbb{R}^2$ when given an appropriate topology and smooth structure. As such, it is diffeomorphic to $\mathbb{R}^2$. But it is not an embedded submnfd of $\mathbb{R}^2$, because it does not have the subspace top.; see ES9E5(a).

Exercise → Let M be a smooth mnfd and let $S \subseteq M$ be an immersed submnfd. Show that every subset of S that is open in the subspace top. is also open in its given submnfd top.; and the converse is true iff S is embedded.

Given a smooth submnfd that is only known to be immersed, it is often useful to have simple criteria that guarantee that it is embedded. The next proposition gives several such criteria.

· **PROP. 5.15**: Let M be a smooth mnfd and let S be an immersed submnfd of M. If any of the following holds, then S is embedded.

(a) $\operatorname{codim}_M S = 0$.

(b) The inclusion map $\iota : S \hookrightarrow M$ is proper.

(c) S is compact.

PROOF: ES9E1; follows readily from PROP. 4.6. ∎

Although many immersed submnfds are not embedded, the next proposition shows that the local structure of an immersed submnfd is the same as that of an embedded one.

· <u>PROP. 5.16</u> (Immersed submanifolds are locally embedded): If M is a smooth mnfd and $S \subseteq M$ is an immersed submnfd, then for each $p \in S$ there exists a neighborhood U of p in S that is an embedded submnfd of M.

<u>PROOF</u>: By assumption, $\iota: S \hookrightarrow M$ is a smooth immersion, so by the local embedding thm E57E3 every $p \in S$ has a neighborhood U in S s.t. $\iota|_U : U \hookrightarrow M$ is a smooth embedding, which proves the assertion. ∎

Finally, we discuss the tangent space to submnfds. If S is a submnfd of $\mathbb{R}^n$, we intuitively think of the tangent space T_pS at a pt $p \in S$ as a subspace of the tangent space $T_p\mathbb{R}^n$. Similarly, the tangent space to a smooth submnfd of an abstract smooth mnfd can be viewed as a subspace of the tangent space to the ambient mnfd, once we make appropriate identifications.

Let M be a smooth mnfd and let S be an immersed or embedded submnfd of M. Since the inclusion map $\iota: S \hookrightarrow M$ is (at least) a smooth immersion, at each pt $p \in S$ we have an injective linear map $d\iota_p : T_pS \hookrightarrow T_pM$. In terms of derivations, this injection works in the following way: for any vector $v \in T_pS$, the image vector $\tilde{v} = d\iota_p(v) \in T_pM$ acts on smooth fncts on M by

$$\tilde{v}f = d\iota_p(v)(f) = v(f \circ \iota) = v(f|_S).$$

We usually identify T_pS with its image $d\iota_p(T_pS)$ under $d\iota_p$, thereby thinking of T_pS as a certain linear subspace of T_pM. This identification makes sense regardless of whether S is embedded or immersed.

There are several alternative ways of characterizing T_pS as a subspace of T_pM; see ES9E3 and ES9E4 for such results. The next proposition, for instance, gives a useful way to characterize T_pS in the embedded case; one can show that it fails in the non-embedded case.

· <u>PROP. 5.17</u>: Let M be a smooth mnfd, let $S \subseteq M$ be an embedded submnfd and let $p \in S$. As a subspace of T_pM, the tangent space T_pS is characterized by

$$T_pS = \{ v \in T_pM \mid vf = 0 \text{ whenever } f \in C^\infty(M) \text{ with } f|_S = 0 \}.$$

<u>PROOF</u>: Pick $v \in T_pS \subseteq T_pM$. Then $v = d\iota_p(w)$ for some $w \in T_pS$, where $\iota : S \hookrightarrow M$ is the inclusion map. If $f \in C^\infty(M)$ with $f|_S = 0$, then $vf = d\iota_p(w)(f) = w(f|_S) = 0$.

Conversely, if $v \in T_pM$ satisfies $vf = 0$ whenever f vanishes on S, w.h.t.s.t. $v = d\iota_p(w)$ for some $w \in T_pS$. Let $(x^1, \ldots, x^n)$ be slice coordinates for S in some neighborhood U of p, so that

$$U \cap S = \{ (x^1, \ldots, x^n) \in U \mid x^{k+1} = \ldots = x^n = 0 \},$$

and $(x^1, \ldots, x^k)$ are coordinates for $U \cap S$. Since the inclusion map $\iota : U \cap S \hookrightarrow M$ has the coordinate representation

$$\iota(x^1, \ldots, x^k) = (x^1, \ldots, x^k, 0, \ldots, 0)$$

in these coordinates, it follows that $T_pS \cong d\iota_p(T_pS)$ is exactly the subspace of T_pM spanned by

$$\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^k}\Big|_p.$$

If we write the coordinate representation of $v \in T_pM$ as

$$v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\Big|_p,$$

then $v \in T_pS$ iff $v^j = 0$ for all $j > k$.

Let φ be a smooth bump fnct supported in U that is equal to 1 in a neighborhood of p. Choose an index $j > k$ and consider the fnct $f(x) = \varphi(x)x^j$, extended to be zero on $M \setminus \operatorname{supp}\varphi$. Then f vanishes identically on S, so

$$0 = vf = \sum_{i=1}^{n} v^i \frac{\partial(\varphi(x)x^j)}{\partial x^i}(p) \overset{\text{product rule}}{\underset{\text{+ properties}}{=}} v^j.$$

Thus, $v \in T_pS$, as desired. ∎

Given a smooth mnfd M and a subset S of M, there are two very different questions one can ask. The simplest question is whether S is an embedded submnfd. Since embedded submnfds are exactly those subsets satisfying the local slice condition, this is simply a question about the subset S itself: either it is an embedded submnfd or it is not, and if so, then the topology and smooth structure making it into an embedded submnfd are uniquely determined according to ES8E6.

A more subtle question is whether S can be an immersed submnfd. In this case, neither the topology nor the smooth structure is known in advance, so one needs to ask whether there are any topology and smooth structure on S making it into an immersed submnfd. This question is not always straightforward to answer, and it can be especially tricky to prove that S is not an immersed submnfd. Here is an example of how this can be done.

<u>EXAMPLE 5.18</u>: Consider the subset

$$S = \{(x,y) \in \mathbb{R}^2 \mid y = |x|\} \subseteq \mathbb{R}^2 .$$

It is easy to check that $S \setminus \{(0,0)\}$ is an embedded 1-dim submnfd of $\mathbb{R}^2$, so if S itself is an immersed submnfd at all, it must be 1-dim. Suppose there were some smooth mnfd structure on S making it into an immersed submnfd. Then $T_{(0,0)}S$ would be a 1-dim subspace of $T_{(0,0)}\mathbb{R}^2$, so by ES9E3(a) there would be a smooth curve $\gamma: (-\varepsilon,\varepsilon) \to \mathbb{R}^2$ whose image is in S, and that satisfies $\gamma(0) = (0,0)$ and $\gamma'(0) \neq 0$. Writing $\gamma(t) = (x(t), y(t))$, we see that $y(t)$ takes a global minimum at $t=0$, so $y'(0) = 0$. On the other hand, since every pt $(x,y) \in S$ satisfies $x^2 = y^2$, we have $x^2(t) = y^2(t)$ for all $t \in (-\varepsilon,\varepsilon)$. Differentiating twice and setting $t=0$, we conclude that $2x'(0)^2 = 2y'(0)^2 = 0$, which is a contradiction. Thus, there is no such smooth mnfd structure on S.

ADDENDUM: SARD'S THM AND WHITNEY'S THMS

· THM (Sard's thm): If $F: M \to N$ is a smooth map between smooth mnfds, then the set of critical values of F has measure zero in N.

$\rightsquigarrow$ "almost all" $c \in N$ are regular values of F $\Rightarrow$

$\Rightarrow$ "almost all" level sets $F^{-1}(c)$ of F are properly embedded submnfds of M of dimension $\dim M - \dim N$.

· THM (Whitney's embedding thm): Every smooth n-mnfd admits a proper smooth embedding into $\mathbb{R}^{2n+1}$.

$\rightsquigarrow$ Every smooth n-mnfd is diffeomorphic to a properly embedded submnfd of $\mathbb{R}^{2n+1}$

(use Whitney's embedding thm, PROP. 5.3, Claim 3 from the proof of PROP. 4.6 and ES8E1(b))

· THM (Whitney's immersion thm): Every smooth n-mnfd admits a smooth immersion into $\mathbb{R}^{2n}$.

The above two thms are sometimes referred to as the easy or weak Whitney embedding and immersion thms, because Whitney obtained later the following improvements.

· THM (Strong Whitney embedding thm): Given $n \geq 1$, every smooth n-mnfd admits a smooth embedding into $\mathbb{R}^{2n}$.

· THM (Strong Whitney immersion thm): Given $n \geq 2$, every smooth n-mnfd admits a smooth immersion into $\mathbb{R}^{2n-1}$.

For the proofs of all the above results, as well as a discussion of sets of measure zero (in $\mathbb{R}^n$ or in smooth mnfds) we refer to [Lee, Chapter 6 and Appendix C].