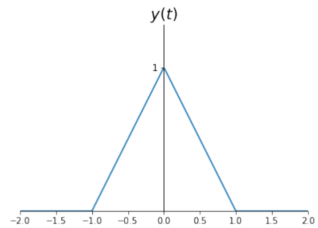
# **Final Exam** - January 27, 2023

Last Name: ........................ First Name: ......................... Sciper: ..........

## [10 pts] Exercise 1

Given the following time domain signal: $y(t) = \begin{cases} 1 - |t| & \text{if } t \in [-1, 1], \\ 0 & \text{otherwise.} \end{cases}$



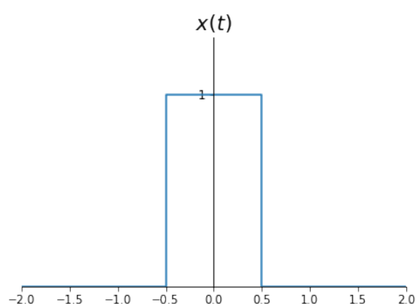

1. **[4 pts]** Find the time domain signal $x(t)$ such that $y(t) = (x * x)(t)$.

2. **[2 pts]** Compute the Fourier transform $Y(F) = \mathcal{F}\{y(t)\}$.

3. **[4 pts]** Let's now consider the frequency domain signal $Z(F) = \frac{\sin^3(\pi F)}{\pi^2 F^2} = \frac{\sin^2(\pi F)}{\pi^2 F^2}. \sin(\pi F)$. Compute its reverse Fourier transform $z(t) = \mathcal{F}^{-1}\{Z(F)\}$ **and** express it as a weighted sum of two time-shifted version of $y(t)$.

## Solution 1

1. The time domain signal $x(t)$ such that $y(t) = (x * x)(t)$ is:




$$x(t) = \begin{cases} 1 & \text{if } t \in [-1/2, 1/2], \\ 0 & \text{otherwise.} \end{cases}$$

Indeed,

$$
\begin{aligned}
(x * x)(t) &= \int_{-\infty}^{+\infty} x(\tau)x(t-\tau)\mathrm{d}\tau \\
&= \int_{-\frac{1}{2}}^{+\frac{1}{2}} x(t-\tau)\mathrm{d}\tau \\
&= \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} x(\tilde{\tau})\mathrm{d}\tilde{\tau}
\end{aligned}
$$

where $\tilde{\tau} = t - \tau$. It follows that:

$$
(x * x)(t) = \begin{cases} 0 & \text{if } t < -1, \\ 1+t & \text{if } t \in [-1, 0], \\ 1-t & \text{if } t \in ]0, 1], \\ 0 & \text{if } t > 1. \end{cases}
$$

And therefore it holds that $y(t) = (x * x)(t)\ \forall t \in \mathbb{R}$.

2. Using the above property, we obtain:

$$
\begin{aligned}
Y(F) &= \mathcal{F}\{y(t)\} \\
&= \mathcal{F}\{(x * x)(t)\} \\
&= \mathcal{F}\{x(t)\}^2 \\
&= \mathrm{sinc}^2(F)
\end{aligned}
$$

3. Using the time shifting property of the Fourier transform, we find:

$$
\begin{aligned}
z(t) &= \mathcal{F}^{-1}\{Z(F)\}(t) \\
&= \mathcal{F}^{-1}\left\{\frac{\sin^3(\pi F)}{\pi^2 F^2}\right\}(t) \\
&= \mathcal{F}^{-1}\left\{\mathrm{sinc}^2(F)\sin(\pi F)\right\}(t) \\
&= \mathcal{F}^{-1}\left\{\mathrm{sinc}^2(F)\left(\frac{e^{j\pi F} - e^{-j\pi F}}{2j}\right)\right\}(t) \\
&= \mathcal{F}^{-1}\left\{\frac{\mathrm{sinc}^2(F)}{2j}e^{j\pi F}\right\}(t) - \mathcal{F}^{-1}\left\{\frac{\mathrm{sinc}^2(F)}{2j}e^{-j\pi F}\right\}(t) \\
&= \mathcal{F}^{-1}\left\{\frac{\mathrm{sinc}^2(F)}{2j}\right\}(t+1/2) - \mathcal{F}^{-1}\left\{\frac{\mathrm{sinc}^2(F)}{2j}\right\}(t-1/2) \\
&= \frac{1}{2j}\left[y(t+1/2) - y(t-1/2)\right]
\end{aligned}
$$

# [18 pts] Exercise 2

We investigate the effect of interchanging the order of two operations on a signal, namely, sampling and performing a memory-less[1] non-linear operation.

1. **[10 pts]** Let's consider the two signal-processing systems depicted in Figure 1, where the C/D (Continuous to Discrete) and D/C (Discrete to Continuous) converters are ideal. The memory-less non-linear operation is determined by the mapping: $g[x] = x^2 = x.x$.

[1]An operation is said to be memory-less if its outcome is fully determined by the current value of the input signal.

(a) **[6 pts]** For the two systems depicted in Figure 1, sketch the signal frequency spectra at points ①, ②, and ③ when the sampling frequency is $F_s = 2F_m$ and $x_c(t)$ has the Fourier transform shown in Figure 2.

(b) **[2 pts]** Is $y_1(t) = y_2(t)$? If not. why not? Explain your answer.

(c) **[2 pts]** Is $y_1(t) = x_c^2(t)$? Explain your answer.

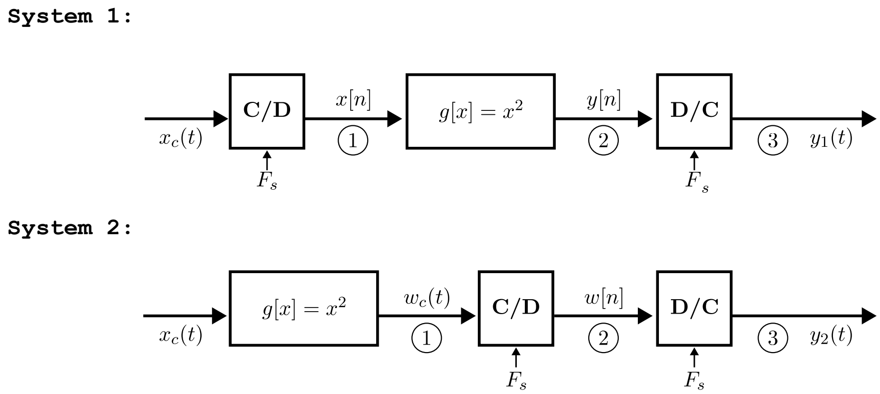



Figure 1: Signal processing systems 1 and 2.

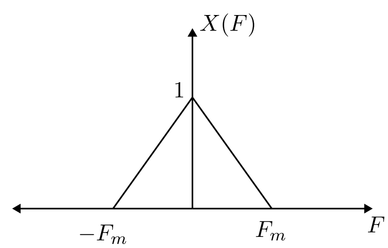



Figure 2: Frequency spectrum $X(F)$ of signal $x_c(t)$.

2. **[4 pts]** We now consider the time domain signal $x_c(t) = A\cos(30\pi t)$ and System 1 only. The sampling frequency is $F_s = 40$ Hz. Sketch again the signal frequency spectra at points ①, ②, and ③. Is $y_1(t) = x_c^2(t)$? Explain why or why not.

3. **[4 pts]** A practical application of using memory-less non-linear devices is that of digitizing a signal having a large dynamic range[2]. Suppose we compress the dynamic range by passing the signal through a memory-less non-linear device prior to C/D (Continuous to Discrete) conversion and then expand its dynamic range using the corresponding inverse non-linear device after C/D conversion.

   For example, consider system 3 of Figure 3 where we compress the dynamic range of a non-negative signal $x_c(t)$ using a non-linear operation $g[x] = \sqrt{x}$ prior to C/D conversion and expand it back with an inverse non-linear operation $g^{-1}[v] = v^2$.

   What is the impact of such non-linear compression operation prior to the C/D converter in the choice of the sampling frequency $F_s$? Is the Nyquist sampling frequency after applying the non-linear operation on the signal less than the original Nyquist sampling frequency? Explain your answer. *Hint:* Discuss how the bandwidth of a signal changes after applying the non-linear compression operation $g[x] = x^{1/M}$ on it. (where $M \in \{2, 3, 4, ..., \infty\}$)

[2] Ratio of the largest measurable signal to the smallest measurable signal

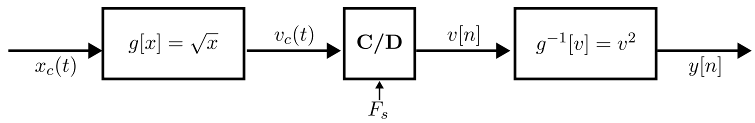



Figure 3: Signal processing system 3.

# Solution 2

1. $y_1(t) = y_2(t)$. Indeed, convolution is a linear process and so is aliasing. Therefore periodic convolution is equivalent to convolution followed by aliasing.
   $y_1(t) \neq x_c^2(t)$. Indeed, the bandwidth of $x_c^2(t)$ is twice that of $y_1(t)$, hence they are not equivalent.

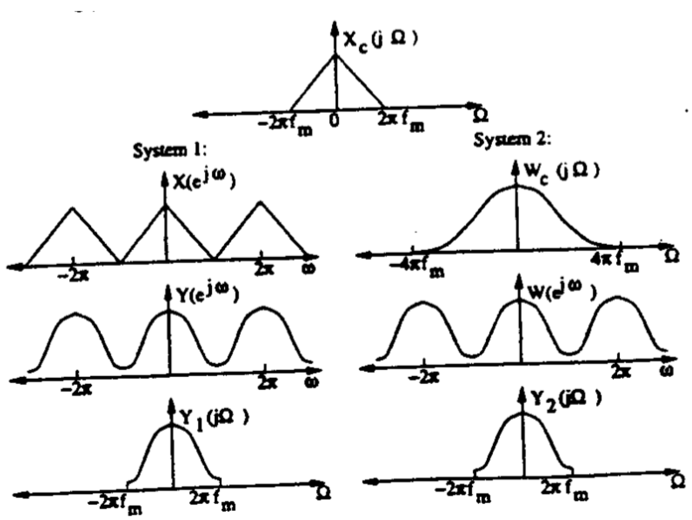
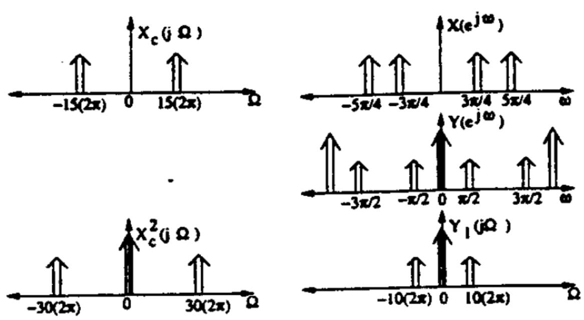



Figure 4: Frequency spectra at points 1, 2 and 3 of systems 1 and 2.

2. By sketching the frequency spectra of $x_c(t)$ and $y_1(t)$, we can clearly see that they are not equivalent:



3. This is the inverse to part (c). Since multiplication in time corresponds to convolution in frequency, a signal $x^2(t)$ has at most two times the bandwidth of $x(t)$. Therefore, $x^{1/2}$ will have at least $\frac{1}{2}$ the bandwidth of $x(t)$. If we run our signal through a box that will raise it to the $1/M$ power, then the sampling rate can be decreased by a factor of $M$.