⇝ proof of IFT: ES6E3
⇝ local diffeomorphisms are discussed in ES6
⇝ $\exists$ smooth embeddings which are neither open nor closed maps

The most important fact about maps of constant rank is the following consequence of the inverse fnct thm, which says that a smooth map of constant rank can be placed locally into a particularly simple canonical form by a change of coordinates. (This is a non-linear version of the canonical form theorem for linear maps; see [Lee, Thm B.20].)

· **THM 4.7** (Rank theorem): Let M and N be smooth mnfds of dim m and n, respectively, and let $F: M \to N$ be a smooth map of constant rank r. For each $p \in M$ there exist smooth charts (U, φ) for M centered at p and (V, ψ) for N centered at $F(p)$ s.t. $F(U) \subseteq V$, in which F has a coordinate representation of the form

$$\hat{F}(x^1, \dots, x^r, x^{r+1}, \dots, x^m) = (x^1, \dots, x^r, 0, \dots, 0).$$

In particular, if F is a smooth submersion, then this becomes

$$\hat{F}(x^1, \dots; x^n, x^{n+1}, \dots, x^m) = (x^1, \dots, x^n),$$

while if F is a smooth immersion, then this becomes

$$\hat{F}(x^1, \dots, x^m) = (x^1, \dots, x^m, 0, \dots, 0).$$

PROOF: Since the thm is local, after choosing smooth coordinates we can replace M and N by open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$. The fact that $DF(p)$ has rank r implies that its matrix has some $r \times r$ submatrix with non-zero determinant. By reordering the coordinates, w.m.a.t. it is the upper left submatrix, $(\partial F^i/\partial x^j)$ for $i,j \in \{1,\dots,r\}$. We relabel the standard coordinates as

$$(x,y) = (x^1,\dots,x^r,y^1,\dots,y^{m-r}) \text{ in } \mathbb{R}^m$$

and

$$(v,w) = (v^1,\dots,v^r,w^1,\dots,w^{n-r}) \text{ in } \mathbb{R}^n.$$

By initial translation of the coordinates, w.l.o.g. w.m.a.t. $p=(0,0)$ and $F(p)=(0,0)$. If we write $F(x,y)=(Q(x,y),R(x,y))$ for some smooth maps $Q: U \to \mathbb{R}^r$ and $R: U \to \mathbb{R}^{n-r}$, then our hypothesis is that $(\partial Q^i/\partial x^j)$ is non-singular at $(0,0)$.

Define

$$\varphi: U \to \mathbb{R}^m, \quad \varphi(x,y) = (Q(x,y),y)$$

and observe that its total derivative at $(0,0)$ is

$$D\varphi(0,0) = \begin{pmatrix} \frac{\partial Q^i}{\partial x^j}(0,0) & \frac{\partial Q^i}{\partial y^j}(0,0) \\ 0 & \delta^i_j \end{pmatrix}, \quad (\delta^i_j \text{ Kronecker delta})$$

which is non-singular by virtue of the hypothesis. Therefore, by the inverse fnct thm, there are connected neighborhoods U_0 of $(0,0)$ and $\tilde{U}_0$ of $\varphi(0,0)=(0,0)$ s.t. $\varphi|_{U_0}: U_0 \to \tilde{U}_0$ is a diffeomorphism. By shrinking U_0 and $\tilde{U}_0$ if necessary, w.m.a.t. $\tilde{U}_0$ is an open cube. Writing the inverse map as

$$\varphi^{-1}(x,y) = (A(x,y),B(x,y))$$

for some smooth fncts $A: \tilde{U}_0 \to \mathbb{R}^r$ and $B: \tilde{U}_0 \to \mathbb{R}^{m-r}$, we compute

$$(x,y) = \varphi\big(A(x,y), B(x,y)\big) = \Big(Q\big(A(x,y), B(x,y)\big), B(x,y)\Big).$$

Comparing y components shows that $B(x,y) = y$, and therefore φ^{-1} has the form

$$\varphi^{-1}(x,y) = \big(A(x,y), y\big).$$

On the other hand, $\varphi \circ \varphi^{-1} = Id$ implies $Q\big(A(x,y), y\big) = x$, and therefore $F \circ \varphi^{-1}$ has the form

$$(F \circ \varphi^{-1})(x,y) \overset{P.50}{=\!=} \big(x, \widetilde{R}(x,y)\big),$$

where $\widetilde{R} : \widetilde{U}_0 \to \mathbb{R}^{n-r}$ is defined by $\widetilde{R}(x,y) = R\big(A(x,y), y\big)$. The Jacobian matrix of $F \circ \varphi^{-1}$ at an arbitrary pt $(x,y) \in \widetilde{U}_0$ is

$$D\big(F \circ \varphi^{-1}\big)(x,y) = \begin{pmatrix} \delta^i_j & 0 \\ \dfrac{\partial \widetilde{R}^i}{\partial x^j}(x,y) & \dfrac{\partial \widetilde{R}^i}{\partial y^j}(x,y) \end{pmatrix}.$$

Since composing with a diffeomorphism does not change the rank of a map, the above matrix has rank r everywhere in $\widetilde{U}_0$. The first r columns are obviously linearly independent, so the rank can be r only if $\partial \widetilde{R}^i / \partial y^j$ vanish identically on $\widetilde{U}_0$, which implies that $\widetilde{R}$ is actually independent of $(y^1, \dots, y^{m-r})$. (This is one reason we arranged for $\widetilde{U}_0$ to be a cube.) Thus, if we let $S(x) = \widetilde{R}(x,0)$, then we have

$$(F \circ \varphi^{-1})(x,y) = \big(x, S(x)\big). \qquad (*)$$

To complete the proof, we need to define an appropriate smooth chart in some neighborhood of $(0,0) \in V$. Consider the open subset

$$V_0 := \{(v,w) \in V \mid (v,0) \in \tilde{U}_0\} \subseteq V$$

and note that V_0 is an open neighborhood of $(0,0)$. Since $\tilde{U}_0 \ni (0,0) = \varphi(0,0)$ is a cube and $(F \circ \varphi^{-1})(x,y) = (x, S(x))$, it follows that $(F \circ \varphi^{-1})(\tilde{U}_0) \subseteq V_0$ (because $(v,w) \in \tilde{U}_0 \Rightarrow (F \circ \varphi^{-1})(v,w) = (v, S(v)) \in V$ and $(v,0) \in \tilde{U}_0$ by construction of $\tilde{U}_0$), so $F(U_0) \subseteq V_0$. Define

$$\psi : V_0 \longrightarrow \mathbb{R}^n, \quad \psi(v,w) = (v, w - S(v)).$$

This is an open map and a diffeomorphism onto its image, because its inverse is given explicitly by $\psi^{-1}(s,t) = (s, t + S(s))$. Thus, (V_0, ψ) is a smooth chart. It follows from $(*)$ that

$$(\psi \circ F \circ \varphi^{-1})(x,y) = \psi(x, S(x)) = (x, S(x) - S(x)) = (x, 0),$$

which was to be proved. ∎

The next corollary can be viewed as a more invariant statement of the rank theorem. It says that maps of constant rank are precisely the ones whose local behavior is the same as that of their differentials.

· COR. 4.8: Let $F: M \longrightarrow N$ be a smooth map. Assume that M is connected. Then t.f.a.e.:

(a) For each $p \in M$ there exist smooth charts containing p and $F(p)$ in which the coordinate representation of F is linear.

(b) F has constant rank.

PROOF:

(b) $\Rightarrow$ (a): Follows from the rank theorem.

(a) $\Rightarrow$ (b): Since every linear map has constant rank, it follows that the rank of F is constant in a neighborhood of each pt, and thus by connectedness it is constant on all of M. ■

· <u>THM 4.9</u> (Global rank theorem): Let $F: M \to N$ be a smooth map of constant rank.

(a) If F is surjective, then it is a smooth submersion.

(b) If F is injective, then it is a smooth immersion.

(c) If F is bijective, then it is a diffeomorphism.

<u>PROOF</u>: Assume that $m = \dim M$, $n = \dim N$ and $r = \operatorname{rk} F$.

(a) see [Lee, Thm 4.14(a)].

(b) Assume that F is not a smooth immersion, so that $r < m$. By the rank theorem, for each $p \in M$ we can choose charts around p and $F(p)$ in which F has the coordinate representation $\hat{F}(x^1, \dots, x^r, x^{r+1}, \dots, x^m) = (x^1, \dots, x^r, 0, \dots, 0)$. Thus, $\hat{F}(0, \dots, 0, \varepsilon) = (0, \dots, 0)$ for any $0 < \varepsilon \ll 1$, which shows that F is not injective ↯

(c) F: bijective $\Rightarrow$ F: injective + surjective

$\xRightarrow[(b)]{(a)}$ F: smooth immersion + smooth submersion

$\xRightarrow{\text{ESGE5(a)}}$ F: local diffeomorphism

$\xRightarrow[F\text{: bijective}]{\text{ESGE41(f)}}$ F: diffeomorphism

■

Let $\pi : M \to N$ be a cont. map.

- A <u>section of π</u> is a cont right inverse for π, i.e., a cont. map $\sigma : N \to M$ s.t. $\pi \circ \sigma = Id_N$.
- A <u>local section of π</u> is a cont. map $\sigma : U \to M$ defined on some open subset $U \subseteq N$ and satisfying the analogous relation $\pi \circ \sigma = Id_U$.

Many of the important properties of smooth submersions follow from the fact that they admit an abundance of smooth local sections, which we prove below.

- <u>THM 4.10</u> (Local section theorem): Let $\pi : M \to N$ be a smooth map. Then π is a smooth submersion iff every pt of M is in the image of a smooth local section of π.

<u>PROOF</u>: Set $m := \dim M$ and $n := \dim N$.

"$\Rightarrow$": Fix $p \in M$ and set $q = \pi(p)$. By the rank theorem we can choose smooth coordinates $(x^1, \dots, x^m)$ centered at p and $(y^1, \dots, y^n)$ centered at q in which π has the coordinate representation

$$\pi(x^1, \dots, x^n, x^{n+1}, \dots, x^m) = (x^1, \dots, x^n).$$

If ε is a sufficiently small positive real number, then the coordinate cube

$$C_\varepsilon := \{ x \mid |x^i| < \varepsilon ,\ 1 \le i \le m \}$$

is a neighborhood of p whose image under π is the cube

$$C'_\varepsilon := \{ y \mid |y^i| < \varepsilon ,\ 1 \le i \le n \}.$$

The map $\sigma : C'_\varepsilon \to C_\varepsilon$ given by

$$\sigma(x^1, \dots, x^n) = (x^1, \dots, x^n, 0, \dots, 0)$$

is a smooth local section of π satisfying $\sigma(q) = p$.

"$\Leftarrow$": Given $p \in M$, let $\sigma : U \to M$ be a smooth local section of π s.t. $\sigma(q) = p$, where $q = \pi(\sigma(q)) = \pi(p) \in N$. The equation $\pi \circ \sigma = Id_U$ implies that $d\pi_p \circ d\sigma_q = Id_{T_qN}$ by PROP 3.7(b), which in turn implies that $d\pi_p$ is surjective. Since $p \in M$ was arbitrary, we conclude that π is a smooth submersion. ■

⌈Recall: If X is a top. sp., Y is a set, and $\pi : X \to Y$ is a surj. map, then <u>the quotient top. on Y determined by π</u> is defined by declaring a subset $V \subseteq Y$ to be open if $\pi^{-1}(V)$ is open in X. If X and Y are top. sp., a map $\pi : X \to Y$ is called a <u>quotient map</u> if it is surj. and cont. and Y has the quotient top. determined by π. ⌋

· <u>PROP. 4.11</u>: Let $\pi : M \to N$ be a smooth submersion. Then π is an open map, and if it is surjective, then it is a quotient map.

<u>PROOF</u>: The second assertion follows from the first one (a surj., open, cont. map is a quotient map), so we now prove that π is an open map. Let W be an open subset of M and let $q \in \pi(W)$. For any $p \in W$ s.t. $\pi(p) = q$, by THM 4.10 there is a neighborhood U of q on which there exists a smooth local section $\sigma : U \to M$ of π with $\sigma(q) = p$. For each $y \in \sigma^{-1}(W)$,

the fact that $\sigma(y) \in W$ implies that $y = \pi(\sigma(y)) \in \pi(W)$. Thus, $\sigma^{-1}(W)$ is an open neighborhood of q contained in $\pi(W)$, which implies that $\pi(W)$ is open. ■

The next three theorems provide important tools that are frequently used when studying submersions and demonstrate that surjective smooth submersions play a role in smooth mnfd theory analogous to the role of quotient maps in topology.

· THM 4.12 (Characteristic property of surjective smooth submersions): Let $\pi: M \to N$ be a surjective smooth submersion. For any smooth mnfd P, a map $F: N \to P$ is smooth iff $F \circ \pi : M \to P$ is smooth:

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "F \circ \pi"] & \\ N \arrow[r, "F"'] & P \end{tikzcd}$$

PROOF: If F is smooth, then $F \circ \pi$ is also smooth by PROP. 2.6(d). Conversely, assume that $F \circ \pi$ is smooth and let $q \in N$. Since π is surj., there is $p \in M$ s.t. $\pi(p) = q$, and then THM 4.10 guarantees the existence of a neighborhood U of q in N and a smooth local section $\sigma: U \to M$ of π s.t. $\sigma(q) = p$. Then $\pi \circ \sigma = Id_U$ implies

$$F|_U = F|_U \circ Id_U = F|_U \circ (\pi \circ \sigma) = (F \circ \pi) \circ \sigma,$$

which is a composition of smooth maps. Hence, F is smooth by

PROP. 2.6(d) = ES3E3(e) and ES3E2(a). $\blacksquare$

⇝ ES7E4 explains the sense in which this property is "characteristic"

⇝ ES7E5 shows that the converse of THM 4.12 is false.

· THM 4.13 (Passing smoothly to the quotient): Let $\pi : M \to N$ be a surjective smooth submersion. If P is a smooth mnfd and if $F : M \to P$ is a smooth map that is constant on the fibers of π, then there exists a unique smooth map $\tilde{F} : N \to P$ s.t. $\tilde{F} \circ \pi = F$:

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "F"] & \\ N \arrow[r, dashed, "\tilde{F}"'] & P \end{tikzcd}$$

Try to define $\tilde{F}$ set-theoretically, to prove its uniqueness, and to check its continuity using that π is a quotient map

PROOF: By PROP. 4.11, π is a quotient map, and by [Lee, Theorem A.30] there exists a unique cont. map $\tilde{F} : N \to P$ s.t. $\tilde{F} \circ \pi = F$. This map is smooth by THM 4.12. $\blacksquare$

· THM 4.14 (Uniqueness of smooth quotients): Let $\pi_1 : M \to N_1$ and $\pi_2 : M \to N_2$ be surjective smooth submersions that are constant on each other's fibers. Then there exists a unique diffeomorphism $F : N_1 \to N_2$ s.t. $F \circ \pi_1 = \pi_2$:

$$\begin{tikzcd} & M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ N_1 \arrow[rr, dashed, "F"'] & & N_2 \end{tikzcd}$$

PROOF : ES7E6.