Differential Geometry II - Smooth Manifolds
Winter Term 2023/2024
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Exercise Sheet 7 – Solutions

Exercise 1:

(a) Let N and $M_1, \ldots, M_k$ be smooth manifolds, where $k \geq 2$, and let $F_i \colon N \to M_i$ be smooth maps, where $1 \leq i \leq k$. Show that the map

$$G \colon N \to \prod_{i=1}^{k} M_i,\ x \mapsto \big(F_1(x), \ldots, F_k(x)\big)$$

is smooth and that its differential at any point $p \in N$ is of the form

$$\big(dG_p\big)(v) = \big(d(F_1)_p(v), \ldots, d(F_k)_p(v)\big),\ v \in T_pN.$$

(b) Let M be a smooth manifold. Show that there exists a smooth map $f \colon M \to [0, +\infty)$ that is proper.

[Hint: Use a function of the form $f = \sum_{i=1}^{+\infty} c_i \psi_i$, where $(\psi_i)_{i=1}^{+\infty}$ is a partition of unity and the c_i's are real numbers.]

(c) Let $F \colon M \to N$ be an injective smooth immersion between smooth manifolds. Show that there exists a smooth embedding $G \colon M \to N \times \mathbb{R}$.

[Hint: Use parts (a) and (b).]

Solution:

(a) The fact that G is smooth follows immediately from part (b) of *Exercise* 4*, Sheet* 3, and the fact that the differential of G at $p \in N$ has the above form follows readily from part (b) of *Exercise* 1*, Sheet* 4 and *Exercise* 3*, Sheet* 4.

(b) Let $\mathfrak{U} = (U_i)_{i\in\mathbb{N}}$ be a countable basis for the topology of M such that $\overline{U}_i$ is compact for each $i \in \mathbb{N}$, and let (ψ_i) be a smooth partition of unity subordinate to $\mathfrak{U}$. Consider now a sequence $(c_i)_{i\in\mathbb{N}}$ of non-negative real numbers satisfying $\lim_{i\to\infty} c_i = +\infty$ (for instance, take $c_i = i$) and define the smooth function

$$f \colon M \to \mathbb{R},\ x \mapsto \sum_{i\in\mathbb{N}} c_i\, \psi_i(x).$$

We may view $f(x)$ as a weighted average of the numbers c_i, using the coefficients $\psi_i(x)$ as weights, which satisfy $0 \leq \psi_i(x) \leq 1$ and $\sum_i \psi_i(x) = 1$ for every $x \in M$. In particular, if $I_x \subseteq \mathbb{N}$ is the set of indices i such that U_i contains the point $x \in M$, then any upper or lower bound for the numbers c_i with $i \in I_x$ is also an upper or lower bound for $f(x)$. Thus, if $f(x) < c$ for some $c > 0$, then for any $i \in \mathbb{N}$ such that $\psi_i(x) \neq 0$ (there are only finitely many such indices by construction of a partition of unity) it holds that $c_i < c$, so x is contained in the union of the corresponding (first few) U_i's, since then $x \in \operatorname{supp} \psi_i \subseteq U_i$.

We will now show that f is proper. Let $K \subseteq \mathbb{R}$ be a compact set. Take any number $c > 0$ such that $K \subseteq (-c, c)$ and pick an index $i_c \in \mathbb{N}$ such that $c_i \geq c$ for every $i \geq i_c$. The preimage $f^{-1}(K)$ consists of points $x \in M$ satisfying $f(x) < c$, and is therefore contained in the compact set $\bigcup_{i<i_c} \overline{U}_i$. Since the set $f^{-1}(K)$ is closed, we conclude that it is compact, as desired.

(c) By part (b) there exists a smooth proper function $f\colon M \to \mathbb{R}$. Consider now the map

$$G\colon M \to N \times \mathbb{R},\ x \mapsto \big(F(x), f(x)\big),$$

which is smooth and whose differential has the form $dG = \big(dF, df\big)$ by part (a). Since F is injective, one immediately sees that G is also injective. Moreover, since F is a smooth immersion, and thus its differential dF_p is injective at every point $p \in M$, it follows readily that $dG_p = \big(dF_p, df_p\big)$ is also injective at every point $p \in M$. Consequently, G is an injective smooth immersion.

Next, we claim that G is a proper map. Given a compact subset $K \subseteq N \times \mathbb{R}$, we will show that $G^{-1}(K)$ is a compact subset of M. To this end, since $N \times \mathbb{R}$ is a Hausdorff space, K is in particular a closed subset of $N \times \mathbb{R}$, and since G is continuous, the preimage $G^{-1}(K)$ is a closed subset of M. Now, since the projection to the second factor $\mathrm{pr}_2\colon N \times \mathbb{R} \to \mathbb{R}$ is continuous, the image $\mathrm{pr}_2(K)$ is a compact subset of $\mathbb{R}$, and since f is proper by assumption, the preimage $f^{-1}\big(\mathrm{pr}_2(K)\big)$ is a compact subset of M, which contains the closed set $G^{-1}(K)$. Hence, $G^{-1}(K)$ is a compact subset of M, as claimed.

In conclusion, G is a smooth embedding by the above and by *Proposition 4.6*(b), as asserted.

Exercise 2:

(a) Show that the inclusion map $\iota\colon \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ is a smooth embedding.

(b) Consider the map

$$F\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (2 + \tanh t) \cdot (\cos t, \sin t).$$

(i) Show that F is an injective smooth immersion.

(ii) Show that F is a smooth embedding.

Solution:

(a) Consider the graph coordinates $\big(U_i^{\pm} \cap \mathbb{S}^n, \varphi_i^{\pm}\big)$ for $\mathbb{S}^n$; see *Example 1.8*(2). We have shown in *Example 2.7* that the inclusion map $\iota\colon \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ is smooth, because its coordinate representation with respect to any of the graph coordinates is

$$\widehat{\iota}(u^1, \dots, u^n) = \left(u^1, \dots, u^{i-1}, \pm\sqrt{1 - \|u\|^2}, u^i, \dots, u^n\right),$$

which is smooth on its domain, the unit ball $\mathbb{B}^n = \{u = (u^1, \ldots, u^n) \in \mathbb{R}^n \mid \|u\| < 1\}$. The Jacobian matrix of the coordinate representation $\widehat{\iota} = \iota \circ (\varphi_i^{\pm})^{-1}$ of ι with respect to the graph coordinates has the form

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 \\ \frac{\mp u^1}{\sqrt{1-\|u\|^2}} & \frac{\mp u^2}{\sqrt{1-\|u\|^2}} & \ldots & \frac{\mp u^{i-1}}{\sqrt{1-\|u\|^2}} & \frac{\mp u^i}{\sqrt{1-\|u\|^2}} & \frac{\mp u^{i+1}}{\sqrt{1-\|u\|^2}} & \ldots & \frac{\mp u^{n-1}}{\sqrt{1-\|u\|^2}} & \frac{\mp u^n}{\sqrt{1-\|u\|^2}} \\ 0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

In particular, we observe that each of these $(n+1) \times n$ matrices (which represent the differential of ι in coordinate bases) has rank n. Hence, ι is an injective smooth immersion. Since $\mathbb{S}^n$ is compact, by *Proposition 4.6*(c) we conclude that ι is a smooth embedding.

(b) We first deal with (i). Clearly, F is smooth. Next, recall that the function

$$\|F(t)\| = 2 + \tanh t, \;\; t \in \mathbb{R}$$

is strictly increasing, which implies that F is injective. Finally, to show that F is a smooth immersion, it suffices to show that $F'(t) \neq 0$ for every $t \in \mathbb{R}$. To this end, recall that

$$\frac{d}{dt} \tanh t = \frac{1}{\cosh^2 t}, \;\; t \in \mathbb{R},$$

so we have

$$F'(t) = \left(-(2 + \tanh t) \sin t + \frac{1}{\cosh^2 t} \cos t, \; (2 + \tanh t) \cos t + \frac{1}{\cosh^2 t} \sin t \right), \;\; t \in \mathbb{R},$$

and thus

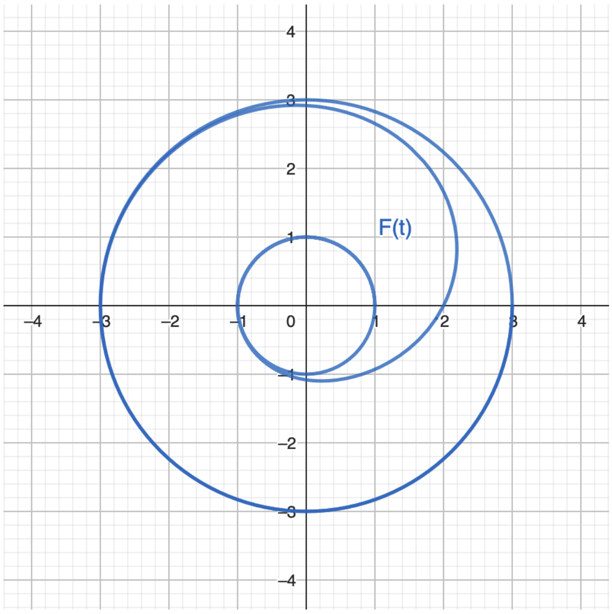
$$\|F'(t)\|^2 = (2 + \tanh t)^2 + \frac{1}{\cosh^4 t} > 0 \;\text{ for all } t \in \mathbb{R},$$

which implies that $F'(t) \neq 0$ for every $t \in \mathbb{R}$, as desired.

We now deal with (ii). Consider the open annulus

$$U := \left\{ x \in \mathbb{R}^2 \mid 1 < \|x\| < 3 \right\} \subseteq \mathbb{R}^2$$

and note that $F(t) \in U$ for every $t \in \mathbb{R}$. (Incidentally, the image of $F|_{[-4\pi, 4\pi]}$ has been plotted below.)



Thus, F may be viewed as an injective smooth immersion $F\colon \mathbb{R} \to U$. Since the inclusion map $\iota\colon U \hookrightarrow \mathbb{R}^2$ is a smooth embedding, in view of part (a)(iii) of *Exercise* 1*, Sheet* 6 and *Proposition 4.6*, to prove (ii), it suffices to show that $F\colon \mathbb{R} \to U$ is a *proper* map. To this end, given a compact subset K of U, we have to show that $F^{-1}(K)$ is a compact subset of $\mathbb{R}$, or equivalently that it is closed and bounded. Since $K \subseteq U$ is compact and $U \subseteq \mathbb{R}^2$ is Hausdorff, K is a closed subset of U, and since F is continuous, $F^{-1}(K)$ is a closed subset of $\mathbb{R}$. Now, denote by m (resp. M) the minimum (resp. the maximum) norm of the points of K, and observe that $[m, M] \subseteq (1, 3)$. Denote also by ℓ (resp. L) the preimage of m (resp. M) under the strictly increasing function

$$g\colon \mathbb{R} \to (1,3),\ t \mapsto \|F(t)\| = 2 + \tanh t$$

and note that $F^{-1}(K) \subseteq [\ell, L]$, which shows that $F^{-1}(K)$ is a bounded subset of $\mathbb{R}$. This finishes the proof of (ii).

Exercise 3 (*Local embedding theorem*)**:** Let $F\colon M \to N$ be a smooth map between smooth manifolds. Show that F is a smooth immersion if and only if every point in M has a neighborhood U such that $F|_U\colon U \to N$ is a smooth embedding.

[Hint: Use the rank theorem and the closed map lemma.]

Solution: If every point in M has a neighborhood on which F is a smooth embedding, then F has full rank everywhere, so it is a smooth immersion.

Conversely, assume that F is a smooth immersion, and let $p \in M$. We first claim that p has a neighborhood on which F is injective. Indeed, by the *rank theorem* there is an open neighborhood U_1 of p on which F has a coordinate representation of the form

$$\widehat{F}(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0),$$

and thus $F|_{U_1}$ is injective. Now, consider a precompact neighborhood U of p such that $\overline{U} \subseteq U_1$. The restriction of F to $\overline{U}_1$ is an injective continuous map with compact domain and Hausdorff codomain, so it is a topological embedding; see *Claims 1* and *2* in the proof of *Proposition 4.6*. Since any restriction of a topological embedding is again a topological embedding, $F|_U$ is both a topological embedding and a smooth immersion, and hence $F|_U$ is a smooth embedding.

Exercise 4: Let M and N be smooth manifolds, and let $\pi\colon M \to N$ be a surjective smooth submersion. Show that there is no other smooth manifold structure on N that satisfies the conclusion of *Theorem 4.12*; in other words, assuming that $\widetilde{N}$ represents the same set as N with a possibly different topology and smooth structure, and that for every smooth manifold P, a map $F\colon \widetilde{N} \to P$ is smooth if and only if $F \circ \pi$ is smooth, show that Id_N is a diffeomorphism between N and $\widetilde{N}$.

Solution: Denote by Id_N, respectively $\mathrm{Id}_{\widetilde{N}}$, the identity map of N, respectively $\widetilde{N}$, with the smooth structure of N, respectively $\widetilde{N}$, on both the source and the target. Denote also by $\mathrm{Id}_{N,\widetilde{N}}$, respectively $\mathrm{Id}_{\widetilde{N},N}$, the identity map, where on the source, respectively on the target, we put the smooth structure of N, and where on the target, respectively on the source, we put the smooth structure of $\widetilde{N}$. In addition, denote by π_N, respectively $\pi_{\widetilde{N}}$, the surjective smooth submersion with the smooth structure of N, respectively of $\widetilde{N}$, on the target. Now, note that

$$\mathrm{Id}_{N,\widetilde{N}} \circ \pi_N = \pi_{\widetilde{N}},$$

which is smooth, so by the assumption on N applied to $P = \widetilde{N}$ and $F = \mathrm{Id}_{N,\widetilde{N}}$ we conclude that $\mathrm{Id}_{N,\widetilde{N}}$ is smooth. On the other hand, we also have

$$\mathrm{Id}_{\widetilde{N},N} \circ \pi_{\widetilde{N}} = \pi_N,$$

which is smooth, so by the assumption on $\widetilde{N}$ applied to $P = N$ and $F = \mathrm{Id}_{\widetilde{N},N}$ we conclude that $\mathrm{Id}_{\widetilde{N},N}$ is smooth. Hence, $\mathrm{Id}_{N,\widetilde{N}}$ is a diffeomorphism with inverse $\mathrm{Id}_{\widetilde{N},N}$.

Exercise 5: Consider the map

$$\pi\colon \mathbb{R}^2 \to \mathbb{R},\ (x,y) \mapsto xy.$$

Show that π is surjective and smooth, and that for each smooth manifold P, a map $F\colon \mathbb{R} \to P$ is smooth if and only if $F \circ \pi$ is smooth; but π is not a smooth submersion. (Therefore, the converse of *Theorem 4.12* is false.)

Solution: Both the smoothness and the surjectivity of π is clear. Therefore, if a map $F\colon \mathbb{R} \to P$ is smooth, then $F \circ \pi$ is also smooth by *Exercise* 3*, Sheet* 3. Now, assume that we have a smooth manifold P and a map of sets $F\colon \mathbb{R} \to P$ such that $F \circ \pi$ is smooth. Consider the map

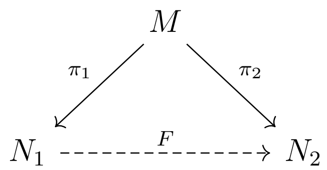
$$\iota\colon \mathbb{R} \to \mathbb{R}^2,\ x \mapsto (x,1),$$

which is clearly smooth and additionally satisfies $\pi \circ \iota = \mathrm{Id}_{\mathbb{R}}$. Hence, the map

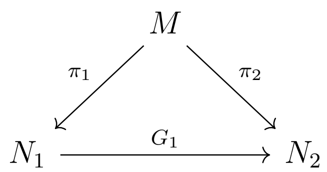
$$F = F \circ \mathrm{Id}_{\mathbb{R}} = (F \circ \pi) \circ \iota$$

is smooth. Finally, note that the Jacobian of π is given by $(y\ x)$, which vanishes at $(x,y) = 0$, so π is not a smooth submersion.

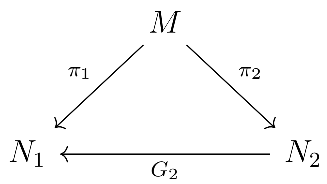
Exercise 6 (*Uniqueness of smooth quotients*)**:** Let $\pi_1 \colon M \to N_1$ and $\pi_2 \colon M \to N_2$ be surjective smooth submersions that are constant on each other's fibers. Show that there exists a unique diffeomorphism $F \colon N_1 \to N_2$ such that $F \circ \pi_1 = \pi_2$:



Solution: Since π_1 is a surjective smooth submersion and since π_2 is constant on the fibers of π_1, by *Theorem 4.13* there exists a unique smooth map $G_1 \colon N_1 \to N_2$ such that $G_1 \circ \pi_1 = \pi_2$:



By reversing now the roles of π_1 and π_2, we see that there exists a unique smooth map $G_2 \colon N_2 \to N_1$ such that $G_2 \circ \pi_2 = \pi_1$:



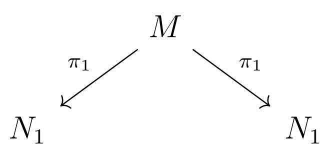
We thus obtain the identities

$$G_2 \circ G_1 \circ \pi_1 = \pi_1 \tag{$*$}$$

and

$$G_1 \circ G_2 \circ \pi_2 = \pi_2. \tag{$**$}$$

Considering the diagram



and observing that $\mathrm{Id}_{N_1} \circ \pi_1 = \pi_1$, we deduce by (the uniqueness part of) *Theorem 4.13* and $(*)$ that

$$G_2 \circ G_1 = \mathrm{Id}_{N_1} .$$

Considering now the corresponding diagram for π_2 and using $(**)$ instead, we infer similarly that

$$G_1 \circ G_2 = \mathrm{Id}_{N_2} .$$

Hence, $F \coloneqq G_1 \colon N_1 \to N_2$ is a diffeomorphism such that $F \circ \pi_1 = \pi_2$, which is unique (with this property) by construction.