Differential Geometry II - Smooth Manifolds
Winter Term 2023/2024
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Exercise Sheet 5 – Solutions

Exercise 1:

(a) Let (x, y) denote the standard coordinates on $\mathbb{R}^2$. Verify that $(\widetilde{x}, \widetilde{y})$ are smooth global coordinates on $\mathbb{R}^2$, where

$$\widetilde{x} = x \quad \text{and} \quad \widetilde{y} = y + x^3.$$

Let p be the point $(1, 0) \in \mathbb{R}^2$ (in standard coordinates), and show that

$$\left.\frac{\partial}{\partial x}\right|_p \neq \left.\frac{\partial}{\partial \widetilde{x}}\right|_p,$$

even though the coordinate functions x and $\widetilde{x}$ are identically equal.

(This shows that each coordinate vector $\partial/\partial x^i|_p$ depends on the entire coordinate system, not just on the single coordinate function x^i.)

(b) *Polar coordinates on* $\mathbb{R}^2$: Consider the map

$$\begin{aligned} \Phi \colon W := (0, +\infty) \times (-\pi, \pi) &\to \mathbb{R}^2 \\ (r, \theta) &\mapsto (r\cos\theta,\, r\sin\theta). \end{aligned}$$

(i) Show that Φ is a diffeomorphism onto its image $U := \Phi(W)$.
(Therefore, Φ^{-1} can be considered as a smooth chart on $\mathbb{R}^2$, and it is common to call its component functions the *polar coordinates* (r, θ) on $\mathbb{R}^2$.)

(ii) Let p be a point in $\mathbb{R}^2$ whose polar coordinate representation is $(r, \theta) = (2, \pi/2)$, and let $v \in T_p\mathbb{R}^2$ be the tangent vector whose polar coordinate representation is

$$v = 3\left.\frac{\partial}{\partial r}\right|_p - \left.\frac{\partial}{\partial \theta}\right|_p.$$

Compute the coordinate representation of v in terms of the standard coordinate vectors

$$\left.\frac{\partial}{\partial x}\right|_p, \left.\frac{\partial}{\partial y}\right|_p.$$

(c) *Spherical coordinates on* $\mathbb{R}^3$: Consider the map

$$\begin{aligned}\Psi\colon W := (0,+\infty)\times(-\pi,\pi)\times(0,\pi) &\to \mathbb{R}^3\\ (r,\varphi,\theta) &\mapsto (r\cos\varphi\sin\theta,\ r\sin\varphi\sin\theta,\ r\cos\theta).\end{aligned}$$

(i) Show that Ψ is a diffeomorphism onto its image $U := \Psi(W)$.
(Therefore, Ψ^{-1} can be considered as a smooth chart on $\mathbb{R}^3$, and it is common to call its component functions the *spherical coordinates* (r,φ,θ) on $\mathbb{R}^3$.)

(ii) Express the coordinate vectors

$$\left.\frac{\partial}{\partial r}\right|_p,\ \left.\frac{\partial}{\partial \varphi}\right|_p,\ \left.\frac{\partial}{\partial \theta}\right|_p$$

of this chart at some point $p \in U$ in terms of the standard coordinate vectors

$$\left.\frac{\partial}{\partial x}\right|_p,\ \left.\frac{\partial}{\partial y}\right|_p,\ \left.\frac{\partial}{\partial z}\right|_p.$$

Solution:

(a) Consider the function

$$\psi\colon \mathbb{R}^2 \to \mathbb{R}^2,\ (x,y)\mapsto (x, y+x^3).$$

Observe that ψ is smooth and bijective with inverse function

$$\psi^{-1}\colon \mathbb{R}^2 \to \mathbb{R}^2,\ (\widetilde{x},\widetilde{y})\mapsto \left(\widetilde{x}, \widetilde{y}-\widetilde{x}^3\right),$$

which is also smooth. Hence, ψ is a global smooth coordinate chart on $\mathbb{R}^2$; in other words, its components $(\widetilde{x},\widetilde{y})$ are smooth global coordinates on $\mathbb{R}^2$.

We have

$$\frac{\partial \widetilde{x}}{\partial x}(x,y) = 1 \quad\text{and}\quad \frac{\partial \widetilde{y}}{\partial x}(x,y) = 3x^2,$$

and hence

$$\left.\frac{\partial}{\partial x}\right|_p = 1\cdot\left.\frac{\partial}{\partial \widetilde{x}}\right|_p + 3\cdot\left.\frac{\partial}{\partial \widetilde{y}}\right|_p \neq \left.\frac{\partial}{\partial \widetilde{x}}\right|_p.$$

(b) We deal with (i) and (ii) separately.

(i) Geometrically, $r\in(0,+\infty)$ is the distance from the origin, and $\theta\in(-\pi,\pi)$ is the angle from the negative x-axis. Observe now that the image of Φ is the plane without the non-positive x-axis, that is,

$$U = \Phi(W) = \left\{(x,y)\in\mathbb{R}^2 \mid x>0\right\}.$$

Note also that $\Phi\colon W\to U$ is bijective with inverse

$$\Phi^{-1}\colon U\to W,\ (x,y)\mapsto\left(\sqrt{x^2+y^2},\ \arctan\frac{y}{x}\right).$$

Since both Φ and Φ^{-1} are clearly smooth, we conclude that $\Phi\colon W\to U$ is a diffeomorphism.

(ii) We have

$$\left.\frac{\partial}{\partial r}\right|_p = \cos\left(\frac{\pi}{2}\right)\left.\frac{\partial}{\partial x}\right|_p + \sin\left(\frac{\pi}{2}\right)\left.\frac{\partial}{\partial y}\right|_p = \left.\frac{\partial}{\partial y}\right|_p$$

and

$$\left.\frac{\partial}{\partial \theta}\right|_p = -2\sin\left(\frac{\pi}{2}\right)\left.\frac{\partial}{\partial x}\right|_p + 2\cos\left(\frac{\pi}{2}\right)\left.\frac{\partial}{\partial y}\right|_p = -2\left.\frac{\partial}{\partial x}\right|_p,$$

so v has the following coordinate representation in standard coordinates:

$$v = 2\left.\frac{\partial}{\partial x}\right|_p + 3\left.\frac{\partial}{\partial y}\right|_p.$$

(c) We deal with (i) and (ii) separately.

(i) Geometrically, $r \in (0, +\infty)$ is the distance from the origin, $\varphi \in (-\pi, \pi)$ is the angle from the $x < 0$ half of the (x, z)-plane, and $\theta \in (0, \pi)$ is the angle from the positive z-axis. Observe now that the image of Ψ is the 3-dimensional space without the z-axis and the non-positive x-axis, that is,

$$U = \Psi(W) = \mathbb{R}^3 \setminus \Big(\big\{(0,0,z) \in \mathbb{R}^3 \mid z \in \mathbb{R}\big\} \cup \big\{(x,0,0) \in \mathbb{R}^3 \mid x \leq 0\big\}\Big),$$

and also that $\Psi\colon W \to U$ is bijective. Furthermore, Ψ is clearly smooth with Jacobian matrix

$$J_\Psi = \begin{pmatrix} \cos\varphi\sin\theta & -r\sin\varphi\sin\theta & r\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & r\cos\varphi\sin\theta & r\sin\varphi\cos\theta \\ \cos\theta & 0 & -r\sin\theta \end{pmatrix}$$

and Jacobian determinant

$$\det J_\Psi = -r^2\sin\theta,$$

which does not vanish for any $(r, \theta) \in (0, +\infty) \times (0, \pi)$. Hence, Ψ is a local diffeomorphism by the *inverse function theorem*, and since it is bijective, it is actually a diffeomorphism.

(ii) Since

$$r = (x^2 + y^2 + z^2)^{1/2}$$

and

$$r\sin\theta = (x^2 + y^2)^{1/2},$$

we have

$$\begin{aligned}
\left.\frac{\partial}{\partial r}\right|_p &= \left.\frac{\partial x}{\partial r}\frac{\partial}{\partial x}\right|_p + \left.\frac{\partial y}{\partial r}\frac{\partial}{\partial y}\right|_p + \left.\frac{\partial z}{\partial r}\frac{\partial}{\partial z}\right|_p \\
&= \cos\varphi\sin\theta\left.\frac{\partial}{\partial x}\right|_p + \sin\varphi\sin\theta\left.\frac{\partial}{\partial y}\right|_p + \cos\theta\left.\frac{\partial}{\partial z}\right|_p \\
&= \frac{1}{(x^2 + y^2 + z^2)^{1/2}}\left(x\left.\frac{\partial}{\partial x}\right|_p + y\left.\frac{\partial}{\partial y}\right|_p + z\left.\frac{\partial}{\partial z}\right|_p\right),
\end{aligned}$$

$$\begin{aligned}\left.\frac{\partial}{\partial\varphi}\right|_p &= \frac{\partial x}{\partial\varphi}\left.\frac{\partial}{\partial x}\right|_p + \frac{\partial y}{\partial\varphi}\left.\frac{\partial}{\partial y}\right|_p + \frac{\partial z}{\partial\varphi}\left.\frac{\partial}{\partial z}\right|_p \\ &= -r\sin\varphi\sin\theta\left.\frac{\partial}{\partial x}\right|_p + r\cos\varphi\sin\theta\left.\frac{\partial}{\partial y}\right|_p \\ &= -y\left.\frac{\partial}{\partial x}\right|_p + x\left.\frac{\partial}{\partial y}\right|_p,\end{aligned}$$

and

$$\begin{aligned}\left.\frac{\partial}{\partial\theta}\right|_p &= \frac{\partial x}{\partial\theta}\left.\frac{\partial}{\partial x}\right|_p + \frac{\partial y}{\partial\theta}\left.\frac{\partial}{\partial y}\right|_p + \frac{\partial z}{\partial\theta}\left.\frac{\partial}{\partial z}\right|_p \\ &= r\cos\varphi\cos\theta\left.\frac{\partial}{\partial x}\right|_p + r\sin\varphi\cos\theta\left.\frac{\partial}{\partial y}\right|_p - r\sin\theta\left.\frac{\partial}{\partial z}\right|_p \\ &= \frac{xz}{(x^2+y^2)^{1/2}}\left.\frac{\partial}{\partial x}\right|_p + \frac{yz}{(x^2+y^2)^{1/2}}\left.\frac{\partial}{\partial y}\right|_p - (x^2+y^2)^{1/2}\left.\frac{\partial}{\partial z}\right|_p.\end{aligned}$$

Exercise 2: Consider the inclusion $\iota\colon \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$, where both $\mathbb{S}^2$ and $\mathbb{R}^3$ are endowed with the standard smooth structure. Let $p = (p^1, p^2, p^3) \in \mathbb{S}^2$ with $p^3 > 0$. What is the image of the differential $d\iota_p\colon T_p\mathbb{S}^2 \to T_p\mathbb{R}^3$?

Solution: Observe that the given point $p \in \mathbb{S}^2$ is contained in the domain of the smooth chart (U_3^+, φ_3^+) for $\mathbb{S}^2$, where

$$U_3^+ = \left\{(x^1, x^2, x^3) \in \mathbb{R}^3 \mid x^3 > 0\right\}$$

and

$$\varphi_3^+ : U_3^+ \cap \mathbb{S}^2 \to \mathbb{B}^2,\ (x^1, x^2, x^3) \mapsto (x^1, x^2)$$

with coordinate functions φ^1 and φ^2 (defined in the obvious manner), and recall that the inverse of φ_3^+ is the map

$$(\varphi_3^+)^{-1}\colon \mathbb{B}^2 \to U_3^+ \cap \mathbb{S}^2,\ (u^1, u^2) \mapsto \left(u^1, u^2, \sqrt{1-(u^1)^2-(u^2)^2}\right),$$

see *Example 1.3*(2) and *Example 1.8*(2). Therefore, the coordinate representation $\widehat{\iota}$ of $\iota\colon \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ with respect to the charts (U_3^+, φ_3^+) and $(\mathbb{R}^3, \mathrm{Id}_{\mathbb{R}^3})$ is the function

$$\widehat{\iota}(u^1, u^2) = \left(\mathrm{Id}_{\mathbb{R}^3} \circ \iota \circ (\varphi_3^+)^{-1}\right)(u^1, u^2) = \left(u^1, u^2, \sqrt{1-(u^1)^2-(u^2)^2}\right),$$

and the coordinate representation $\widehat{p}$ of $p \in \mathbb{S}^2$ is $\widehat{p} = \varphi(p) = (p^1, p^2)$. Since the Jacobian matrix of $\widehat{\iota}$, given by

$$J(u^1, u^2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{u^1}{\sqrt{1-(u^1)^2-(u^2)^2}} & -\frac{u^2}{\sqrt{1-(u^1)^2-(u^2)^2}} \end{pmatrix},$$

represents $d\iota_p\colon T_p\mathbb{S}^2 \to T_p\mathbb{R}^3$ in the coordinate bases

$$\left\{\left.\frac{\partial}{\partial\varphi^1}\right|_p, \left.\frac{\partial}{\partial\varphi^2}\right|_p\right\} \subseteq T_p\mathbb{S}^2 \quad \text{and} \quad \left\{\left.\frac{\partial}{\partial x^1}\right|_p, \left.\frac{\partial}{\partial x^2}\right|_p, \left.\frac{\partial}{\partial x^3}\right|_p\right\} \subseteq T_p\mathbb{R}^3,$$

we deduce that

$$\begin{aligned} d\iota_p\left(\frac{\partial}{\partial \varphi^1}\bigg|_p\right) &= 1\cdot\frac{\partial}{\partial x^1}\bigg|_p + 0\cdot\frac{\partial}{\partial x^2}\bigg|_p - \frac{p^1}{\sqrt{1-(p^1)^2+(p^2)^2}}\cdot\frac{\partial}{\partial x^3}\bigg|_p \\ &= \frac{\partial}{\partial x^1}\bigg|_p - \frac{p^1}{p^3}\frac{\partial}{\partial x^3}\bigg|_p \end{aligned}$$

and

$$\begin{aligned} d\iota_p\left(\frac{\partial}{\partial \varphi^2}\bigg|_p\right) &= 0\cdot\frac{\partial}{\partial x^1}\bigg|_p + 1\cdot\frac{\partial}{\partial x^2}\bigg|_p - \frac{p^2}{\sqrt{1-(p^1)^2+(p^2)^2}}\cdot\frac{\partial}{\partial x^3}\bigg|_p \\ &= \frac{\partial}{\partial x^2}\bigg|_p - \frac{p^2}{p^3}\frac{\partial}{\partial x^3}\bigg|_p . \end{aligned}$$

Thus, the image of $d\iota_p$ is the $\mathbb{R}$-vector space spanned by the above two vectors, which can be identified with the vectors $(1,0,-\frac{p^1}{p^3})$ and $(0,1,-\frac{p^2}{p^3})$, respectively, in $\mathbb{R}^3$. It is now easy to check that this 2-dimensional $\mathbb{R}$-vector space is the orthogonal complement of $\langle p\rangle$; namely,

$$d\iota_p\left(T_p\mathbb{S}^2\right) = \langle p\rangle^\perp \cong \left\{v\in\mathbb{R}^3 \mid \langle v,p\rangle = 0\right\}.$$

Exercise 3: Let $M_1,\dots,M_k$ be smooth manifolds. Show that $T(M_1\times\ldots\times M_k)$ is diffeomorphic to $T(M_1)\times\ldots\times T(M_k)$.

Solution: For each $1\le i\le k$, denote by

$$\pi_i\colon M_1\times\ldots\times M_k\to M_i$$

the projection onto the i-the factor. This map is smooth by *Exercise* 4*, Sheet* 3, and hence its global differential

$$d(\pi_i)\colon T(M_1\times\ldots\times M_k)\to TM_i$$

is also a smooth map by *Exercise* 4(a). Again by *Exercise* 4*, Sheet* 3 we thus obtain a smooth map

$$\alpha\colon T(M_1\times\ldots\times M_k)\to TM_1\times\ldots\times TM_k$$

given by $\alpha = \big(d(\pi_1),\dots,d(\pi_k)\big)$. Note that if $p=(p_1,\dots,p_k)\in M_1\times\ldots\times M_k$, then α restricted to the fiber $T_p(M_1\times\ldots\times M_k)$ is just the map defined in *Exercise* 3*, Sheet* 4, so it is in particular an isomorphism. Therefore, α is bijective. It remains to show that α is a diffeomorphism.

To this end, for every $1\le i\le k$, let $(U_i,(x_i^{j_i})_{j_i})$ be a smooth chart for M_i, and denote by $\mathrm{pr}_i\colon TM_i\to M_i$ the projection. By construction of the tangent bundle, $\big(\mathrm{pr}_i^{-1}U_i,(x_i^{j_i})_{j_i},(v_i^{j_i})_{j_i}\big)$ is a smooth coordinate chart, where $(v_i^{j_i})_{j_i}$ are the coordinates of a point $(p,v)\in TM_i$ (with $p\in U_i$) in terms of the basis $\left(\frac{\partial}{\partial x_i^{j_i}}\Big|_p\right)_{j_i}$ of T_pM_i. This yields the chart

$$\big(\mathrm{pr}_1^{-1}U_1\times\ldots\times\mathrm{pr}_k^{-1}U_k,((x_1^{j_1})_{j_1},(v_1^{j_1})_{j_1}),\dots,((x_k^{j_k})_{j_k},(v_k^{j_k})_{j_k})\big)$$

for $T(M_1) \times \ldots \times T(M_k)$. On the other hand, if we denote by

$$\mathrm{pr}\colon T(M_1 \times \ldots \times M_k) \to M_1 \times \ldots \times M_k$$

the projection, then this also yields the chart

$$\big(\mathrm{pr}^{-1}(U_1 \times \ldots \times U_k), (x_i^{j_i})_{ij_i}, (v_i^{j_i})_{ij_i}\big)$$

for $T(M_1 \times \ldots \times M_k)$. In terms of these charts, the map α is just given by

$$\big((x_i^{j_i})_{ij_i}, (v_i^{j_i})_{ij_i}\big) \mapsto \big((x_1^{j_1})_{j_1}, (v_1^{j_1})_{j_1}, \ldots, (x_k^{j_k})_{j_k}, (v_k^{j_k})_{j_k}\big)$$

which is clearly a diffeomorphism. Hence α is locally a diffeomorphism, and as it is bijective, it is actually a diffeomorphism; see part (f) of *Exercise* 4*, Sheet* 6.

Exercise 4:

(a) Let $F\colon M \to N$ be a smooth map. Show that its *global differential* $dF\colon TM \to TN$ (which is just the map whose restriction to each tangent space $T_pM \subseteq TM$ is dF_p) is also a smooth map.

(b) Let $F\colon M \to N$ and $G\colon N \to P$ be smooth maps. Prove the following assertions:

  (i) $d(G \circ F) = dG \circ dF\colon TM \to TP$.

  (ii) $d(\mathrm{Id}_M) = \mathrm{Id}_{TM}\colon TM \to TM$.

  (iii) If F is a diffeomorphism, then $dF\colon TM \to TN$ is also a diffeomorphism, and it holds that $(dF)^{-1} = d(F^{-1})$.

Solution:

(a) Using the local expression for dF_p in coordinates,

$$dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \frac{\partial \widehat{F}^j}{\partial x^i}(\widehat{p})\left.\frac{\partial}{\partial y^j}\right|_{F(p)},$$

we see that dF has the following coordinate representation in terms of natural coordinates for TM and TN:

$$\begin{aligned}
\big(\widetilde{\psi} \circ dF \circ \widetilde{\varphi}^{-1}\big)(x^1, \ldots, x^n, v^1, \ldots, v^n) &= \big(\widetilde{\psi} \circ dF\big)\left(v^i \left.\frac{\partial}{\partial x^i}\right|_{\varphi^{-1}(x)}\right) \\
&= \widetilde{\psi}\left(v^i \frac{\partial \widehat{F}^j}{\partial x^i}\left.\frac{\partial}{\partial y^j}\right|_{F \circ \varphi^{-1}(x)}\right) \\
&= \left(\widehat{F}^1(x), \ldots, \widehat{F}^n(x), \frac{\partial \widehat{F}^1}{\partial x^i}(x)v^i, \ldots, \frac{\partial \widehat{F}^n}{\partial x^i}(x)v^i\right).
\end{aligned}$$

Since F is smooth, and thus its coordinate representation $\widehat{F} = \psi \circ F \circ \varphi^{-1}$ is smooth, the above coordinate representation of dF is smooth, and hence dF is smooth, as claimed.

(b) All assertions follow immediately from *Exercise* 1*, Sheet* 4.

Exercise 5:

(a) Let $f\colon X \to S$ be a map from a topological space X to a set S. Show that if X is connected and if f is *locally constant*, i.e., for every $x \in X$ there exists a neighborhood U of x in X such that $f|_U\colon U \to S$ is constant, then f is constant.

[Hint: Show that f is continuous when S is endowed with the discrete topology.]

(b) Let M and N be smooth manifolds and let $F\colon M \to N$ be a smooth map. Assume that M is connected. Show that $dF_p\colon T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$ if and only if F is constant.

[Hint: Use (a). You may also use (without proof) the fact that any topological manifold is locally (path) connected.]

Solution:

(a) We endow S with the discrete topology, and we claim that $f\colon X \to S$ is continuous. Since then the singletons in S are open, to prove the claim, it suffices to show that the fibers of f are open subsets of X. Fix $s \in S$ and pick $x \in f^{-1}(s)$. Since f is locally constant, there exists an open neighborhood U of x in X such that $f|_U\colon U \to S$ is constant, so for every $u \in U$ we have $f(u) = f(x) = s$, and hence $u \in f^{-1}(s)$. Therefore, the open neighborhood U of x is contained in the fiber $f^{-1}(s)$, i.e., $x \in U \subseteq f^{-1}(s)$. Since $x \in f^{-1}(s)$ was arbitrary, $f^{-1}(s)$ is an open subset of X, and since $s \in S$ was arbitrary, we conclude that f is continuous.

Since S is endowed with the discrete topology, every singleton in S is also closed, and thus every fiber of f is also closed, since f is continuous. In other words, the fibers of f are both closed and open subsets of X, which is a connected space by assumption, and hence each one of them is either empty or the whole space X. It follows that f is constant.

(b) Assume first that F is constant and let $p \in M$. For every $f \in C^\infty(N)$, the composite map $f \circ F\colon M \to \mathbb{R}$ is constant, and hence for every $v \in T_pM$, by *Lemma 3.5* we have $dF_p(v)(f) = v(f \circ F) = 0$. In conclusion, dF_p is the zero linear transformation for every $p \in M$.

Assume now that dF_p is the zero map for each $p \in M$. By assumption and by (a), to prove that F is constant, it suffices to show that F is locally constant. Fix $p \in M$. Since F is smooth, there are smooth charts (U, φ) for M containing p and (V, ψ) for N containing $F(p)$ such that $F(U) \subseteq V$ and the composite map $\widehat{F} = \psi \circ F \circ \varphi^{-1}$ is smooth. By shrinking U if necessary, we may assume that U is connected, and thus $\varphi(U)$ is also connected. Now, for each $q \in U$ we know that the differential dF_q is represented in coordinate bases by the Jacobian matrix of $\widehat{F}$. Since $dF_q = \mathbb{O}$ for every $q \in U$ by assumption, we infer that

$$\frac{\partial \widehat{F}^j}{\partial x^i}(\widehat{q}) = 0 \quad \text{for every } i, \text{ every } j, \text{ and every } \widehat{q} = \varphi(q) \in \varphi(U),$$

and hence $\widehat{F}$ is constant on $\varphi(U)$. It follows that $F = \varphi \circ \widehat{F} \circ \psi^{-1}$ is constant on U. Since $p \in M$ was arbitrary, we conclude that F is locally constant, as desired.