Solution 6
Quantum Information Processing

Exercise 1 *Useful identity for the realisation of CNOT*

Here all matrices that must be exponentiated are diagonal. In this case it is easier to do computations in components.

- $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is diagonal so

$$R_1 = R_2 = \begin{pmatrix} \exp\left(i\frac{\pi}{4}\right) & 0 \\ 0 & \exp\left(-i\frac{\pi}{4}\right) \end{pmatrix} = e^{i\frac{\pi}{4}} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}.$$

- The Hadamard gate is $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

- For the Hamiltonian we have

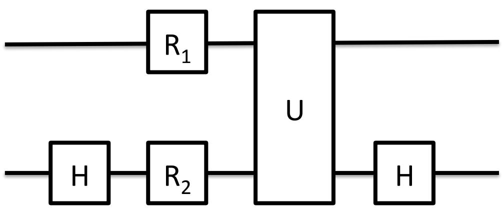
$$\mathcal{H} = \hbar J \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \hbar J \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- If we let evolve for a time $t = \frac{\pi}{4J}$ we find

$$U = \exp\left(-\frac{it}{\hbar}\mathcal{H}\right) = \exp\left(-\frac{i\pi}{4J\hbar}\mathcal{H}\right) = \begin{pmatrix} \exp\left(-i\frac{\pi}{4}\right) & 0 & 0 & 0 \\ 0 & \exp\left(i\frac{\pi}{4}\right) & 0 & 0 \\ 0 & 0 & \exp\left(i\frac{\pi}{4}\right) & 0 \\ 0 & 0 & 0 & \exp\left(-i\frac{\pi}{4}\right) \end{pmatrix},$$

$$\Rightarrow U = \exp\left\{-i\frac{\pi}{4}\right\}. \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Remark: The product of matrices corresponds to the circuit:



The input state enters on the left side $|\psi\rangle$ and the output state is produced on the right $(I_{2\times 2}\otimes H)U(R_1\otimes R_2)(I_{2\times 2}\otimes H)|\psi\rangle$.

We compute the product:

$$
\begin{aligned}
R_1\otimes R_2 &= e^{i\frac{\pi}{2}}\begin{pmatrix}1 & 0\\ 0 & -i\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\ 0 & -i\end{pmatrix}\\
&= e^{i\frac{\pi}{2}}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -i & 0 & 0\\ 0 & 0 & -i & 0\\ 0 & 0 & 0 & -1\end{pmatrix},
\end{aligned}
$$

and

$$
\begin{aligned}
U(R_1\otimes R_2) &= e^{i\frac{\pi}{4}}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & i & 0 & 0\\ 0 & 0 & i & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -i & 0 & 0\\ 0 & 0 & -i & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\\
&= e^{i\frac{\pi}{4}}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}.
\end{aligned}
$$

Also

$$
I_{2\times 2}\otimes H = \frac{1}{\sqrt{2}}\left(\begin{array}{cc|cc}1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ \hline 0 & 0 & 1 & 1\\ 0 & 0 & 1 & -1\end{array}\right)
$$

and

$$
\frac{1}{\sqrt{2}}\left(\begin{array}{cc|cc}1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ \hline 0 & 0 & 1 & 1\\ 0 & 0 & 1 & -1\end{array}\right)\left(\begin{array}{cc|cc}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ \hline 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{array}\right) = \frac{1}{\sqrt{2}}\left(\begin{array}{cc|cc}1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ \hline 0 & 0 & 1 & 1\\ 0 & 0 & 1 & -1\end{array}\right),
$$

and then

$$
\frac{1}{\sqrt{2}}\left(\begin{array}{cc|cc}1 & -1 & 0 & 0\\ 1 & -1 & 0 & 0\\ \hline 0 & 0 & 1 & 1\\ 0 & 0 & 1 & -1\end{array}\right)\frac{1}{\sqrt{2}}\left(\begin{array}{cc|cc}1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ \hline 0 & 0 & 1 & 1\\ 0 & 0 & 1 & -1\end{array}\right) = \frac{1}{2}\left(\begin{array}{cc|cc}2 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ \hline 0 & 0 & 0 & 2\\ 0 & 0 & 2 & 0\end{array}\right),
$$

finally we find

$$(I_{2\times 2} \otimes H)U(R_1 \otimes R_2)(I_{2\times 2} \otimes H) = e^{i\frac{\pi}{4}} \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

This matrix is the CNOT gate up to a global phase (check from the definition!)

Remark: the global phase cannot be measured and has no physical significance. We also note that the CNOT can also be written as (check!)

$$|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes \sigma_x$$

where I is the 2x2 identity matrix.

Exercise 2 *Refocusing*

a) One could write matrices in component form and multiply them. More simply we can apply the identity to computational basis states and check the equality (this means the equality is valid for any vector by linearity).

   For example we for $|\psi_0\rangle = |\uparrow\uparrow\rangle$, we find (using that R_1 flips a spin; verify !)

$$\begin{aligned}
|\psi_1\rangle &= e^{-i\frac{t}{2}\frac{\mathcal{H}}{\hbar}} |\uparrow\uparrow\rangle = e^{-itJ} |\psi_0\rangle , \\
|\psi_2\rangle &= (R_1 \otimes I) |\psi_1\rangle = e^{-itJ} |\downarrow\uparrow\rangle , \\
|\psi_3\rangle &= e^{-i\frac{t}{2}\frac{\mathcal{H}}{\hbar}} |\psi_2\rangle = e^{-itJ} e^{-i\frac{t}{2}\frac{\mathcal{H}}{\hbar}} |\downarrow\uparrow\rangle = e^{-itJ} e^{itJ} |\downarrow\uparrow\rangle = |\downarrow\uparrow\rangle , \\
|\psi_4\rangle &= (R_1 \otimes I) |\psi_3\rangle = (R_1 \otimes I) |\downarrow\uparrow\rangle = |\uparrow\uparrow\rangle ,
\end{aligned}$$

   which shows $|\psi_4\rangle = |\psi_0\rangle = (I_1 \otimes I_2) |\psi_0\rangle$. For other basis states we proceed with similar verifications $|\uparrow\downarrow\rangle , |\downarrow\uparrow\rangle , |\downarrow\downarrow\rangle$.

b) Imagine an initial state. The left hand side of the identity produces some final state. This identity says that the initial and final states are equal. This has a nice application if we operate on qubits. Indeed a pure Heisenberg evolution for a time t (say) will change the state. However it may happen that we want the state to remain the same after time t and that at the same time the Heisenberg evolution cannot be switched of (for example if it given by a natural magnetic interaction between nuclei). One way to achieve our goal (approximately at least) is to interject two short rotation pulses at times $t/2$ and t. Since $J << 1$ the time scale of the two qubit Heisenberg evolution ($1/J >> 1$) is much slower than these short rotation pulses and can be neglected during short intervals when these pulses operate. Therefore the global physical process is approximately described by the left-hand side and is thus approximately equal to the identity matrix.