Homework 3 Solution
Traitement Quantique de l'Information

Exercise 1 *Polarization observable and measurement principle*

1) We first check that $\langle\alpha|\alpha\rangle = \langle\alpha_\perp|\alpha_\perp\rangle = 1$ and $\langle\alpha|\alpha_\perp\rangle = \langle\alpha_\perp|\alpha\rangle = 0$. Therefore, we have

$$\begin{aligned}
\Pi_\alpha^2 &= |\alpha\rangle\langle\alpha|\alpha\rangle\langle\alpha| = |\alpha\rangle\langle\alpha| = \Pi_\alpha \\
\Pi_{\alpha_\perp}^2 &= |\alpha_\perp\rangle\langle\alpha_\perp|\alpha_\perp\rangle\langle\alpha_\perp| = |\alpha_\perp\rangle\langle\alpha_\perp| = \Pi_{\alpha_\perp} \\
\Pi_\alpha\Pi_{\alpha_\perp} &= |\alpha\rangle\langle\alpha|\alpha_\perp\rangle\langle\alpha_\perp| = 0 \\
\Pi_{\alpha_\perp}\Pi_\alpha &= |\alpha_\perp\rangle\langle\alpha_\perp|\alpha\rangle\langle\alpha| = 0
\end{aligned}$$

2)

$$\begin{aligned}
|\langle\theta|\alpha\rangle|^2 &= \langle\theta|\alpha\rangle\langle\theta|\alpha\rangle^* = \langle\theta|\alpha\rangle\langle\alpha|\theta\rangle = \langle\theta|\Pi_\alpha|\theta\rangle, \\
|\langle\theta|\alpha_\perp\rangle|^2 &= \langle\theta|\alpha_\perp\rangle\langle\theta|\alpha_\perp\rangle^* = \langle\theta|\alpha_\perp\rangle\langle\alpha_\perp|\theta\rangle = \langle\theta|\Pi_{\alpha_\perp}|\theta\rangle
\end{aligned}$$

3) The probabilities are

$$\begin{aligned}
\text{Prob}(p = +1) &= |\langle\alpha|\theta\rangle|^2 = |\cos\alpha\cos\theta + \sin\alpha\sin\theta|^2 = (\cos(\theta-\alpha))^2 \\
\text{Prob}(p = -1) &= |\langle\alpha_\perp|\theta\rangle|^2 = |-\sin\alpha\cos\theta + \cos\alpha\sin\theta|^2 = (\sin(\theta-\alpha))^2
\end{aligned}$$

and they sum to 1,

$$\text{Prob}(p = +1) + \text{Prob}(p = -1) = (\cos(\theta-\alpha))^2 + (\sin(\theta-\alpha))^2 = 1$$

4) The expectation is

$$\begin{aligned}
\text{E}[p] &= (+1)\text{Prob}(p = +1) + (-1)\text{Prob}(p = -1) \\
&= (\cos(\theta-\alpha))^2 - (\sin(\theta-\alpha))^2 \\
&= \cos\big(2(\theta-\alpha)\big)
\end{aligned}$$

and the variance is

$$\begin{aligned}
\text{Var}(p) &= \text{E}[p^2] - (\text{E}[p])^2 \\
&= 1 - (\text{E}[p])^2 \\
&= (\cos(\theta-\alpha))^2 - (\sin(\theta-\alpha))^2 \\
&= 1 - \big(\cos(2(\theta-\alpha))\big)^2 \\
&= \big(\sin(2(\theta-\alpha))\big)^2
\end{aligned}$$

In fact they should match with the computation in Dirac notation because

$$
\begin{aligned}
\langle\theta| P_\alpha |\theta\rangle &= \langle\theta| \big((+1)\Pi_\alpha + (-1)\Pi_{\alpha_\perp}\big) |\theta\rangle \\
&= (+1) \langle\theta| \Pi_\alpha |\theta\rangle + (-1) \langle\theta| \Pi_{\alpha_\perp} |\theta\rangle \\
&= (+1)\mathrm{Prob}(p = +1) + (-1)\mathrm{Prob}(p = -1) \\
&= \mathrm{E}[p]
\end{aligned}
$$

and

$$
\begin{aligned}
\langle\theta| P_\alpha^2 |\theta\rangle &= \langle\theta| \big((+1)\Pi_\alpha + (-1)\Pi_{\alpha_\perp}\big)^2 |\theta\rangle \\
&= \langle\theta| \big(\Pi_\alpha^2 - \Pi_\alpha\Pi_{\alpha_\perp} - \Pi_{\alpha_\perp}\Pi_\alpha + \Pi_{\alpha_\perp}^2\big) |\theta\rangle \\
&= \langle\theta| \big(\Pi_\alpha + \Pi_{\alpha_\perp}\big) |\theta\rangle \\
&= (+1)^2 \langle\theta| \Pi_\alpha |\theta\rangle + (-1)^2 \langle\theta| \Pi_{\alpha_\perp} |\theta\rangle \\
&= (+1)^2\mathrm{Prob}(p = +1) + (-1)^2\mathrm{Prob}(p = -1) \\
&= \mathrm{E}[p^2]
\end{aligned}
$$

thereby giving $\mathrm{E}[p] = \langle\theta| P_\alpha |\theta\rangle$ and $\mathrm{Var}(p) = \mathrm{E}[p^2] - (\mathrm{E}[p])^2 = \langle\theta| P_\alpha^2 |\theta\rangle - \langle\theta| P_\alpha |\theta\rangle^2$.

Exercise 2 *Product versus entangled states*

We use the conventional correspondance $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$. Then we have

$$
(\alpha |0\rangle + \beta |1\rangle) \otimes (x |0\rangle + y |1\rangle) = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \begin{pmatrix}x & y\end{pmatrix} = \begin{pmatrix}\alpha x & \alpha y\\ \beta x & \beta y\end{pmatrix}
$$

Thus a product state of two qubits is a rank-one matrix. So for a general state $|\psi\rangle = \sum_{ij} \alpha_{ij} |ij\rangle$, a simple condition is to check if the matrix $A = (\alpha_{ij})_{0\leq i,j\leq 1}$ is of rank one, that is to say (and because $A \neq 0$), if its determinant is 0: $\det(A) = 0$, that is $\alpha_{00}\alpha_{11} = \alpha_{01}\alpha_{10}$

1. product state ($\det(A) = 0$), normalized
2. entangled ($\det(A) = -2$), normalized
3. entangled $\det(A) = -\frac{1}{\sqrt{3\cdot 6}} - \frac{1}{\sqrt{36}} \neq 0$, not normalized
4. all entangled ($\det(A) \neq 0$ for all of them), normalized
5. we find $\det(A) = \epsilon$, so only a product state for $\epsilon = 0$, entangled otherwise, normalized in all cases
6. Let's assume $|\psi\rangle = (x|0\rangle + y|1\rangle) \otimes (u|0\rangle + v|1\rangle) \otimes (s|0\rangle + t|1\rangle)$, then we should have $xut = xvs = yuv = 1$ and all the other products are 0 otherwise. For instance: $yut = 0$. However, because $xut = 1$, then $ut \neq 0$, and because $yuv = 1$ then $y \neq 0$, therefore $yut \neq 0$. So our assumption is wrong and the state is entangled, and also normalized
7. entangled for similar reasons, normalized

8. product state as $|\psi\rangle = \frac{1}{\sqrt{2}^3}\left(|0\rangle + |1\rangle\right)^{\otimes 3}$, and normalized

Exercise 3 *Unitary transformations*

1) The operator is $U = e^{i\omega t}$, thus $U^\dagger = e^{-i\omega t}$ and it is straightforward to check that $U^\dagger U = UU^\dagger = 1$. In this (trivial) case the unitary time evolution matrix is a $1x1$ matrix.

2) We find $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, so in fact $H|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^i|1\rangle)$ for $i \in \{0, 1\}$. Therefore, we have:

$$\langle j| H^\dagger H |i\rangle = \frac{1}{2}\left((\langle 0| + (-1)^j \langle 1|)(|0\rangle + (-1)^i |1\rangle)\right) = \frac{1}{2}\left(1 + (-1)^{i+j}\right) = \delta_{ij}$$

with δ_{ij} the kroenecker symbol.

3) Similarly, we find $X|i\rangle = |i \oplus 1\rangle$ thus $\langle j| X^\dagger X |i\rangle = \langle j \oplus 1 | i \oplus 1\rangle = \delta_{ij}$

4) Step by step:

$$\begin{aligned}
(U_1 \otimes U_2)^\dagger (U_1 \otimes U_2) &= (U_1^\dagger \otimes U_2^\dagger)(U_1 \otimes U_2) & (1)\\
&= (U_1^\dagger U_1) \otimes (U_2^\dagger U_2) & (2)\\
&= I \otimes I & (3)\\
&= I & (4)
\end{aligned}$$

5) It is straightforward to check that: $\mathrm{CNOT}(|i, j\rangle) = |i, j \oplus i\rangle$. Thus:

$$\langle k, l| \mathrm{CNOT}^\dagger \mathrm{CNOT} |i, j\rangle = \langle k, l \oplus k | i, i \oplus j\rangle = \delta_{i,k}\delta_{(l\oplus k),(i\oplus j)} = \delta_{i,k}\delta_{l,j}$$

First let $|\psi_1\rangle = (H|i\rangle) \otimes |j\rangle$, we have using question 2:

$$|\psi_1\rangle = (H|i\rangle) \otimes |j\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^i|1\rangle) \otimes |j\rangle = \frac{1}{\sqrt{2}}(|0, j\rangle + (-1)^i |1, j\rangle)$$

Therefore using question 5:

$$\mathrm{CNOT}|\psi_1\rangle = \frac{1}{\sqrt{2}}\left(|0, j\rangle + (-1)^i |1, j \oplus 1\rangle\right) = |\beta_{ij}\rangle$$

Because H and I are both unitary using question 2, then $U \otimes I$ is unitary using question 4. Then because CNOT is unitary (question 5), using the fact that the set of unitary matrices equipped with the product of matrices is a group, then $O = \mathrm{CNOT} \cdot (H \otimes I)$ is also unitary, hence β_{ij} forms an orthonormal basis as it is the image of an orthonormal basis with the unitary operator O.

Exercise 4 *Interferometer with an atom on the ray*

1) The matrices in Dirac notation are

$$S = \frac{1}{\sqrt{2}} |H\rangle\langle H| + \frac{1}{\sqrt{2}} |H\rangle\langle V| + \frac{1}{\sqrt{2}} |V\rangle\langle H| - \frac{1}{\sqrt{2}} |V\rangle\langle V| + |\text{abs}\rangle\langle\text{abs}|$$
$$R = |H\rangle\langle V| + |V\rangle\langle H| + |\text{abs}\rangle\langle\text{abs}|\,.$$

To find $U = SARS$ we proceed by steps:

$$RS = \frac{1}{\sqrt{2}} |H\rangle\langle H| - \frac{1}{\sqrt{2}} |H\rangle\langle V| + \frac{1}{\sqrt{2}} |V\rangle\langle H| + \frac{1}{\sqrt{2}} |V\rangle\langle V| + |\text{abs}\rangle\langle\text{abs}|\,,$$
$$ARS = |H\rangle\langle\text{abs}| + \frac{1}{\sqrt{2}} |V\rangle\langle H| + \frac{1}{\sqrt{2}} |V\rangle\langle V| + \frac{1}{\sqrt{2}} |\text{abs}\rangle\langle H| - \frac{1}{\sqrt{2}} |\text{abs}\rangle\langle V|$$

and finally

$$\begin{aligned} U = SARS = {} & \frac{1}{2} |H\rangle\langle H| + \frac{1}{2} |H\rangle\langle V| + \frac{1}{\sqrt{2}} |H\rangle\langle\text{abs}| \\ & - \frac{1}{2} |V\rangle\langle H| - \frac{1}{2} |V\rangle\langle V| + \frac{1}{\sqrt{2}} |V\rangle\langle\text{abs}| \\ & + \frac{1}{\sqrt{2}} |\text{abs}\rangle\langle H| - \frac{1}{\sqrt{2}} |\text{abs}\rangle\langle V|\,. \end{aligned}$$

2) As $SARS\,|H\rangle = \frac{1}{2}|H\rangle - \frac{1}{2}|V\rangle + \frac{1}{\sqrt{2}}|\text{abs}\rangle$, the probabilities of the three events are

$$\text{Prob}(D_1) = |\langle V|\,SARS\,|H\rangle|^2 = \frac{1}{4},$$
$$\text{Prob}(D_2) = |\langle H|\,SARS\,|H\rangle|^2 = \frac{1}{4},$$
$$\text{Prob}(\text{abs}) = |\langle\text{abs}|\,SARS\,|H\rangle|^2 = \frac{1}{2},$$

which sum to 1.

3) A legitimate matrix has to be unitary. The first matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

is not unitary because

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq I.$$

The second matrix

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

is unitary because

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

Thus the second matrix may model the absorption and reemission of the photon. Note also that this matrix acts like a Hadamard matrix on the subspace $\{|H\rangle, |\text{abs}\rangle\}$.